Dynamic Networks of Infinite-State Timed Processes

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Abstract. We study dynamic networks of infinite-state timed processes. We model infinite-state processes as unbounded Petri nets. These processes can evolve autonomously, synchronize with each other (e.g., in order to gain access to some shared resources) and be created or become garbage dynamically. We introduce dense time in two different ways. First, we consider that each token in each process carries a real valued clock. We prove that this model can faithfully simulate Turing-complete formalisms and, in particular, safety properties are undecidable for them. Second, we consider locally synchronous processes, where each process carries a single real valued clock. For them, we prove decidability of safety properties by a non-trivial instantiation of the framework of Well-Structured Transition Systems.

1 Introduction

Perhaps the most widely known model of real-time systems is that of Timed Automata [6]. Several tools like UPPAAL or KRONOS are available for them. Natural extensions of Timed Automata are Networks of Timed Automata (NTA) [6] or the Networks of Timed Processes in [3]. Both models consider parameterized systems of finite-state processes, each of which is endowed with a real clock.

Petri nets are one of the best known models for concurrent and distributed systems. Petri nets have been extended with discrete or continuous time in many works [17, 15, 18, 16, 8, 5]. In some, transitions have a duration, while in others they fire atomically in some time interval. They also differ in whether time is considered relative to places, transitions or arcs. In [7] an exhaustive comparison of these models is done, and in particular the class of Petri nets with time relative to arcs is proved to be the most expressive one. Among them, in Timed Petri Nets (*TdPN*) [5] each token is endowed with a real-valued clock. In particular, clocks can be dynamically created or destroyed.

Under the so called counting abstraction, one can think that each token in a place s of a Petri net represents a process in state s. Hence, Petri nets can

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be seen as networks (or products) of finite-state automata. With this intuition in mind, in *TdPN* each (finite-state) process has a real-valued clock. Therefore, they encompass two infinite dimensions: infinitely-many (finite-state) processes and clocks over an infinite (uncountable) domain. In *TdPN* arcs are labeled with intervals. Thus, when a token is taken from a place by a transition the age of the token must be in the interval labeling the corresponding arc, and when a token is put in a place, its age is set to any real value in the corresponding interval. Moreover, it follows the so called weak semantics in timed systems, in which time delays may happen even when they disable transitions.

We extend the work in [5] by allowing each process to be infinite-state in turn. Hence, our model manages infinitely-many timed processes, each of which is infinite-state (a potentially unbounded Petri net). In this way we can for example easily model dynamic networks of processes accessing shared resources, that can be potentially unbounded.

Dynamic process creation is closely related to parametric verification, when the number of processes is a parameter of the system. Indeed, a standard approach for parametric verification is the addition of an "initialization phase" that spawns an unbounded number of processes (see e.g. [10] for a recent discussion). Hence, our results on verifying dynamic systems can also be seen as results on parametric verification of systems with a fixed number of processes.

As a starting point we consider an untimed model we have developed in previous work [19, 20], called *ν-PN*. In *ν-PN* tokens are names, that can be created fresh and matched with other names. Therefore, there can be an unbounded number of different names, each of which can appear an unbounded number of times. Each name can be understood as a process identifier. Hence, *ν-PN* encompass infinitely-many (untimed) processes, each of which can be infinite-state.

We consider two ways in which to introduce time. In the first way we assign a clock to each token in each process. Then, each process is a *TdPN*, that can be created fresh and can synchronize with others. We call this model *Timed ν-PN* (*ν-TdPN*). As in *TdPN*, the age of each token consumed by a transition must belong to a given interval, as well as for the tokens produced by a transition. Moreover, as in *ν-PN*, names can be created and matched. We prove that this model can simulate Turing-complete formalisms and, in particular, even the control-state reachability problem (that of deciding if a given place can be marked) is undecidable.

In the second variant we consider that each process has a single real-valued clock. Since each process is a (concurrent) Petri net, we say these are locally-synchronous processes, and call them *locally synchronous ν-PN* (*ν-lsPN*). Hence, we still encompass infinitely-many processes, each of which is infinite-state and is endowed with a real-valued clock.

For *ν-lsPN* we successfully apply the theory of regions of [3]. More precisely, we prove that working with regions we can give *ν-lsPN* a well-structure, so that they belong to the class of Well-Structured Transition Systems [9, 1], for which the coverability problem is decidable. This proves that control-state reachability (which can be reduced to coverability) is decidable for them. Moreover, safety properties can be reduced to control-state reachability by standard techniques.

Outline: Section 2 gives notations and results we use throughout the paper. Section 3 defines ν-*TdPN* and proves undecidability of control-state reachability for them. In Section 4 we define ν-*lsPN*, and we prove decidability of control-state reachability for them. Finally, in Section 5 we present our conclusions.

2 Preliminaries

Let $\mathbb{N} = \{0, 1, 2, ...\}$ and for each $n \in \mathbb{N}$ let us denote $n^+ = \{1, ..., n\}$ and $n^* = \{0, ..., n\}$. We denote open, closed and mixed intervals of real numbers as (a, c), $[a, b]$ and $[a, c)$ or $(a, b]$, respectively, where $a, b \in \mathbb{N}$ and $c \in \mathbb{N} \cup \{\infty\}$. The set of intervals is denoted by $\mathcal{I}$. Let $\mathbb{R}_{\geq 0} = [0, \infty)$ and for each $x \in \mathbb{R}_{\geq 0}$ we denote by $\lfloor x \rfloor$ and $\mathit{frct}(x)$ the integer and the fractional part of x, respectively.
Well orders: $(X, \leq)$ is a *partial order* (po)[1] if $\leq$ is a reflexive, transitive and antisymmetric binary relation on X. For a po we write $x < y$ iff $x \leq y$ and $x \neq y$. Let $A \subseteq X$. An element $x \in A$ is *minimal* in A if $x' \in A$ with $x' \leq x$ implies $x = x'$. We denote by $\min(A)$ the set of minimal elements in A. The *upward closure* of $A \subseteq X$ is defined as $\uparrow A = \{x \in X \mid \exists x' \in A,\ x' \leq x\}$. We say A is *upward closed* iff $\uparrow A = A$. A po $(X, \leq)$ is a *well partial order* (wpo) if for every infinite sequence $x_0, x_1, \ldots \in X$ there are i and j with $i < j$ such that $x_i \leq x_j$. Equivalently, a po is a wpo iff $\min(U)$ is finite for every upward closed set U. If X is finite, then $(X, =)$ is a wpo. If $(X, \leq_X)$ and $(Y, \leq_Y)$ are wpos, their product $X \times Y$ is well ordered by $(x, y) \leq (x', y')$ iff $x \leq_X x'$ and $y \leq_Y y'$.
Multisets: A (finite) *multiset* m over X is a mapping $m : X \to \mathbb{N}$ with finite support, that is, such that $\mathit{supp}(m) = \{x \in X \mid m(x) > 0\}$ is finite. We denote by $X^{\oplus}$ the set of finite multisets over X. For $m_1, m_2 \in X^{\oplus}$ we define $m_1 + m_2 \in X^{\oplus}$ by $(m_1 + m_2)(x) = m_1(x) + m_2(x)$ and $m_1 \subseteq m_2$ if $m_1(x) \leq m_2(x)$ for every $x \in X$. When $m_1 \subseteq m_2$ we can define $m_2 - m_1 \in X^{\oplus}$ by $(m_2 - m_1)(x) = m_2(x) - m_1(x)$. We denote by $\emptyset$ the empty multiset, that is, $\emptyset(a) = 0$ for every $a \in A$, and $|m| = \sum_{a \in \mathit{supp}(m)} m(a)$. We use set notation for multisets when convenient, with repetitions to account for multiplicities greater than one. Given a po $\leq$ over X, we define the po $\leq^{\oplus}$ over $X^{\oplus}$ as $\{x_1, ..., x_n\} \leq^{\oplus} \{y_1, ..., y_m\}$ if there is an injection $h : n^+ \to m^+$ such that $x_i \leq y_{h(i)}$ for each $i \in n^+$. If $(X, \leq)$ is a wpo then so is $(X^{\oplus}, \leq^{\oplus})$ [12].
Words: Any $u = x_1 \cdots x_n$ with $n \geq 0$ and $x_i \in X$, for all $i \in n^+$, is a (finite) *word* on X. We denote by $X^{\circledast}$ the set of words on X. If $n = 0$ then u is the empty word, denoted by ϵ. If X is a wpo then so is $X^{\circledast}$ [12] ordered by $\leq^{\circledast}$, defined as $x_1 \ldots x_n \leq^{\circledast} y_1 \ldots y_m$ if there is a strictly increasing mapping $h : n^+ \to m^+$ such that $x_i \leq y_{h(i)}$ for each $i \in n^+$.
Transition systems: A *transition system* is a tuple $\mathcal{S} = \langle X, \to, x_0 \rangle$ where X is the set of states, $x_0 \in X$ is the initial state and $\to \subseteq X \times X$ is the transition relation. We write $x \to x'$ instead of $(x, x') \in \to$ and we denote by $\to^*$ the

[1] We only work with po (and not quasi-orders).

reflexive and transitive closure of $\rightarrow$. We say $A \subseteq X$ is *reachable* if $x_0 \rightarrow^* x$ for some $x \in A$. For $x \in X$ we define $Pre(x) = \{x' \mid x' \rightarrow x\}$ and $Pre^*(x) = \{x' \mid x' \rightarrow^* x\}$, and extend them pointwise to sets of states. If X is a po, we can define the *coverability problem*, that of deciding, given U upward closed, whether U is reachable, or equivalently, whether $x_0 \in Pre^*(U)$. All the models in the paper induce transition systems in the obvious way.

ν-Petri Nets: We fix infinite sets Id of names, Var of variables and a subset of special variables $\Upsilon \subset Var$ for fresh name creation. A *ν-Petri Net*(ν-*PN*) [20] is a tuple $N = \langle P, T, F, H \rangle$, where P and T are finite disjoint sets, and $F, H : T \rightarrow (P \times Var)^{\oplus}$ are the input and output functions, respectively. If $(p, x) \in F(t)$ ($(p, x) \in H(t)$), we say that there is an arc from p to t (from t to p) labelled by x.[2] A *marking* is a multiset over $P \times Id$. A *mode* is an injection $\sigma : Var(t) \rightarrow Id$. Modes are extended homeomorphically to $(P \times Var(t))^{\oplus}$. A transition t is *enabled* with mode σ for a marking M if $\sigma(F(t)) \subseteq M$ and for every $\nu \in \Upsilon$, $(p, \sigma(\nu)) \notin M$ for any p. In that case we have $M \xrightarrow{t} M'$, where $M' = (M - \sigma(F(t)) + \sigma(H(t))$ and $M \rightarrow M'$ if $M \xrightarrow{t} M'$ for some $t \in T$. We interpret each name as (the identifier of) a process that can be created, synchronize with other processes or become garbage.

For an example see the second and the third nets in Fig. 1, in which places are represented by circles, transitions by squares, and F and H are represented by labelled arcs. Tokens are represented as names in places. Disregard the intervals in the arcs and the superscripts of the tokens. Transition t can be fired from the marking represented in the second net, reaching the marking in the third one, with mode σ, with $\sigma(x) = a$, $\sigma(y) = b$ and $\sigma(\nu) = c$. In particular, note that the firing of t creates a new name c in place p_4. See [20, 19] for more details.

3 Timed ν-Petri Nets

In this section we define the first extension of ν-*PN* with time, namely Timed ν-Petri nets (ν-*TdPN* for short) and we prove the undecidability of control-state reachability for them.

Basically, a ν-*TdPN* is a ν-PN in which each token has an age. Moreover, arcs are labelled by intervals, meaning that the age of the tokens consumed and produced by the transition must be in these intervals. In Fig. 1 the nets depicted show the same ν-*TdPN* with three different markings, in which tokens are depicted as names with its age as superscript. In the first marking the transition t is not enabled, since the age of the only token in p_2 is not in $[1, 1]$. After a delay of one unit of time, t becomes enabled, and can be fired reaching, for example, the marking depicted in the right.

Let us define ν-*TdPN* formally. Let Var be a set of variables with $\Upsilon \subset Var$.

Definition 1 (Timed ν-Petri Nets). *A* Timed ν-Petri net *is a tuple* $N = \langle P, T, F, H \rangle$*, where:*

[2] We use this notation following [13].

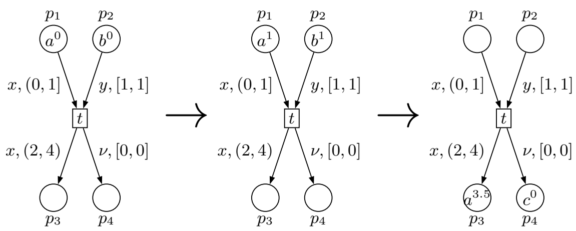

Fig. 1. Firing of transitions in a ν-*TdPN*

- P *and* T *are finite disjoint sets,*
- $F : T \to (P \times \mathit{Var} \times \mathcal{I})^{\oplus}$ *is the input function,*
- $H : T \to (P \times \mathit{Var} \times \mathcal{I})^{\oplus}$ *is the output function.*

For a transition $t \in T$, we take $\mathit{Var}(t)$ as the set of variables adjacent to t, that is, $\mathit{Var}(t) = \{x \in \mathit{Var} \mid \exists p \in P, I \in \mathcal{I},\ (p, x, I) \in F(t)+H(t)\}$. In figures, for each $(p, x, I) \in F(t)$ we draw an arc from p to t, labeled by x, I (and analogously for postconditions).

Definition 2 (Markings). *A* token *of a* ν-*TdPN is an element of* $P \times \mathit{Id} \times \mathbb{R}_{\geq 0}$. *A* marking *is a finite multiset of tokens.*

We write $p(a, r)$ instead of (p, a, r) to denote tokens. Intuitively, $p(a, r)$ is a token in p, carrying the name a, with age $r \geq 0$. We use M, M', M_1,... to range over markings. We say M *marks* $p \in P$ if there are $a \in \mathit{Id}$ and $r \in \mathbb{R}_{\geq 0}$ such that $p(a, r) \in M$. We denote $\mathit{Id}(M) = \{a \mid \exists p, r \text{ with } p(a, r) \in M\}$. We assume $\bullet \in \mathit{Id}$, so that black tokens can appear in markings as in ordinary Petri nets. If an arc is not labeled by any variable we assume that the token involved is $\bullet$. Moreover, in figures we do not write the interval $[0, \infty)$.

Now, let us define the semantics of ν-*TdPN*. Markings may evolve in two different ways: time elapsing or firing of transitions. Time elapsing is acomplished by simply adding the same amount of time to each token in the net. In order to fire a transition $t \in T$, we assign an identifier to each of the variables in $\mathit{Var}(t)$, and we need to ensure that for each $(p, x, I) \in F(t)$, there is a token $p(a, r)$ in the current marking such that $r \in I$.

Definition 3 (Semantics of ν-*TdPN*). Time elapsings*: Given a marking* $M = \{p_1(a_1, r_1), ..., p_n(a_n, r_n)\}$ *and a delay* $d \in \mathbb{R}_{\geq 0}$, *we write* M^{+d} *to denote the marking* $\{p_1(a_1, r_1 + d), ..., p_n(a_n, r_n + d)\}$ *in which the age of all tokens has increased by* d. *Then we write* $M \xrightarrow{d} M^{+d}$.

Firing of transitions*: Let* $t \in T$ *be a transition with* $F(t) = \{p_1(x_1, I_1), ..., p_n(x_n, I_n)\}$ *and* $H(t) = \{q_1(y_1, J_1), ..., q_m(y_m, J_m)\}$. *We say* t *is* enabled *or* can be fired *in marking* M*, evolving to* M'*, and we denote it by* $M \xrightarrow{t} M'$*, if there is an injection* $\sigma : \mathit{Var}(t) \to \mathit{Id}$, $r_1, ..., r_n \in \mathbb{R}_{\geq 0}$ *and* $r'_1, ..., r'_m \in \mathbb{R}_{\geq 0}$ *such that:*

- $r_i \in I_i$ *for any* $i \in n^+$ *and* $r'_j \in J_j$ *for any* $j \in m^+$,
- $\sigma(\nu) \notin \mathit{Id}(M)$ *for any* $\nu \in \Upsilon$,

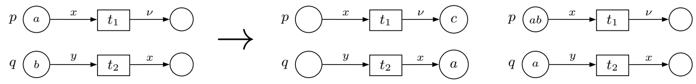

Fig. 2. A *ν-RN* system and the firing of compatible transitions t_1 and t_2, assuming it creates M with $M(p) = \{a, b\}$ and $M(q) = \{a\}$

– $\{p_1(\sigma(x_1), r_1), ..., p_n(\sigma(x_n), r_n)\} \subseteq M$,
– $M' = (M - \{p_1(\sigma(x_1), r_1), ..., p_n(\sigma(x_n), r_n)\}) + \{q_1(\sigma(y_1), r'_1), ..., q_m(\sigma(y_m), r'_m)\}$.

We write $M \to M'$ *if* $M \xrightarrow{t} M'$ *for some* $t \in T$ *or* $M \xrightarrow{d} M'$ *for some* $d \in \mathbb{R}_{\geq 0}$.

Let us prove undecidability of control-state reachability for *ν-TdPN*. Instead of giving a reduction from a well-known Turing-complete model (as Minsky or Turing machines), we first present a Turing complete model based on Petri nets with identifiers, called *ν-RN* systems in [19]. Then we reduce control-state reachability in *ν-RN*, which is undecidable, to our problem. Considering *ν-RN* considerably simplifies our reduction, since the representation gap between both models is certainly smaller than that obtained if we considered a better known Turing-complete formalism.[3]

We briefly present *ν-RN* systems. For more details see [19]. Intuitively, a *ν-RN* system is just a collection of *ν-PN* that can synchronize with each other, and that can create replicas of themselves (hence the name, *ν*-Replicated Nets). For synchronization purposes, we consider a set $\mathcal{L}$ of transition labels.

A *ν-RN system* is a tuple $N = \langle P, T, F, H, \lambda \rangle$, where $\langle P, T, F, H \rangle$ is a *ν-PN* and $\lambda : T \to \mathcal{L}$ labels transitions for two different purposes. On the one hand, it specifies how a transition can be fired: whether it is an autonomous transition, that can be fired in isolation, or a synchronizing transition, that must be fired synchronously with another transition. On the other hand, it indicates which new instances (if any) are created by its firing. An *instance* of N is a multiset over $P \times Id$ (i.e., a marking of the underlying *ν-PN*). A *marking* of N is a multiset of instances of N. Therefore, in *ν-RN* each instance contains tokens, possibly with different names. A synchronous firing can happen whenever two compatible transitions (having labels $s?$ and $s!$) are enabled, according to the enabling condition of *ν-PN*. In that case they can both be fired simultaneously, following the ordinary token game of *ν-PN*. In particular, names can be moved along the nets, be communicated between instances and be created fresh. Moreover, firings may create new instances (see Fig. 2).

The control-state reachability problem for *ν-RN* is that of deciding whether some reachable marking marks a given place. The model of *ν-RN* is Turing-complete [19], and termination for Turing machines can be easily reduced to control-state reachability for *ν-RN*. Hence control-state reachability is undecidable for *ν-RN*.

Proposition 1. *Control-state reachability is undecidable for ν-TdPN.*

[3] Even a reduction from Petri nets with inhibitor arcs, which are also Turing-complete, needs to fill a much bigger representation gap.

Proof (sketch). ν-*RN* systems in which only autonomous transitions may create new instances are powerful enough to simulate Turing machines, as shown in [19]. We reduce control-state reachability for ν-*RN* to the same problem for ν-*TdPN*. The main idea of this proof is how to encode the different instances of a ν-*RN* N by means of a ν-*TdPN* N'. We do it by making all tokens in N' representing the same instance of N having the same age. Then, the firing of transitions in which a single instance takes part, is done by requiring that all tokens involved in the firing are of age exactly 1. Moreover, new instances are created by taken a token from a special place in which we initially put an unbounded amount of tokens, all of them having different ages, at the beginning of the run. Then, the new instance will have the age of this token, which is different to the rest of the ages of token in the net. Synchronizations between several instances are more complicated, but the main idea is the same, namely adding and removing tokens with age 1. Finally, note that the ages of tokens need to be reset to 0 ciclicly. Otherwise, each instance could only participate in one firing per run. Therefore, a mechanism to reset tokens with age 1 to 0 is added. The construction is lossy, in the sense that tokens can get older than 1, thus becoming useless. However, this does not affect control-state reachability. For a detailed version of this proof see Appendix A.

4 Locally synchronous ν-*PN*

In the previous section we have seen that even control-state reachability is undecidable for ν-*TdPN*. Now we define the class of locally synchronous ν-*PN* (ν-*lsPN*), for which each instance has a single clock. ν-*lsPN* can be obtained as a syntactic restriction of ν-*TdPN*, ensuring that each instance uses only one clock. One way to do it is to consider a place in which we store a token of each name in the net, whose age represents the age of the corresponding instance. Formally:

Definition 4. *A ν-lsPN is a ν-TdPN with a distinguished place ages, such that:*

- *If $(p, \nu, I) \in H(t)$ then $(ages, \nu, I) \in H(t)$.*
- *If $(p, x, I) \in F(t) + H(t)$ with $I \neq [0, \infty)$, then $p = ages$.*
- *$(ages, x, I) \in F(t)$ if and only if $(ages, x, I) \in H(t)$. Moreover, there is at most one arc per transition labelled by the same variable leading to ages.*

The first condition ensures that when we create a new instance, we assign it an age. The second condition says that the temporal restrictions are only applied to the tokens representing clocks of the instances. Finally, the third one ensures that there is always a token representing the clock of each instance.

For clarity, we ommit the place *ages* in figures, and we write the temporal restriction in the rest of the arcs instead. In Appendix B. an alternative definition of ν-*lsPN* can be found. In this definition we add read-only arcs to ν-*lsPN*. We use this definition to give the formal prove formally all the results in the Appendix.

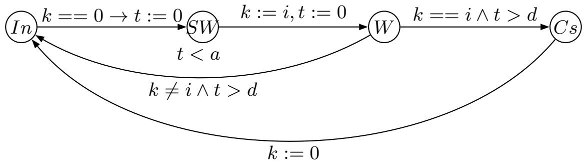

Fig. 3. Timed automaton modelling one process of Fischer's mutual exclusion protocol

Example 1. Fischer's protocol: We model a parameterized version of Fischer's protocol for mutual exclusion, which considers n processes p_i (where n is a parameter), each of those endowed with a real clock x_i. Moreover, a shared integer variable $k \in \{1, \ldots, n\}$ is considered, in order to set the turn for entering the critical section. Each process p_i can be modelled by the timed automaton of Fig.3, and behaves as follows:

```
1  repeat
2    non critical section
3    repeat
4      await k==0;
5      k:=i;
6      delay(d);
7    until k==i;
8    critical section;
9    k:=0;
10   non critical section
11 until false;
```

Process p_i repeatedly tries to enter the critical section. For that purpose, it waits until $k = 0$, which means that no other process is in the critical section. Then, it sets $k = i$, to ask for permission to enter. After a delay of d units of time, if k is still i, the process enters the critical section, setting $k = 0$ after. Otherwise, it repeats lines $4 - 6$. In order to make the algorithm satisfy the mutual exclusion property, it is important to fix a proper delay d, greater than the time it takes each process to execute line 5. We want to guarantee that, for any $n \in \mathbb{N}$,in the system with n processes no two processes are simultaneously in the critical section.

Let us define our model: We consider the net depicted in Fig 4. Intuitively, each token in places In, SW, W and CS represents a different instance, executing lines $1 - 4$, 5, 6 and 8 respectively. The variable k is represented by a place k that contains a black token if $k = 0$ or a token with the identifier that changed its value last.

Notice the transition *new*, that can create any number of processes in their initial state. To prove the previous property, we have to prove that the marking with two tokens in place CS is not coverable.

You can note that the timed automaton in Fig. 3 modelling Fischer's protocol and our model in Fig. 4 are very similar. In [2] Abdulla et al model Fischer's protocol by means of timed Petri nets. As they do not use colors, they need to use the *counting abstraction* (hence considering the state space of each process and the shared variable), so that the obtained model is more complicated than ours.

Notice that the state space of ν-*lsPN* is infinite in various dimensions. It encompasses infinitely-many instances, each of which is potentially unbounded, and contains a clock over an uncountable domain. Moreover, the transition system induced by a ν-*lsPN* is not finitary. Indeed, any marking has infinitely-many successors due to time delays. Next, we use the theory of regions to obtain a finitary transition system over a countable domain. Moreover, this transition system

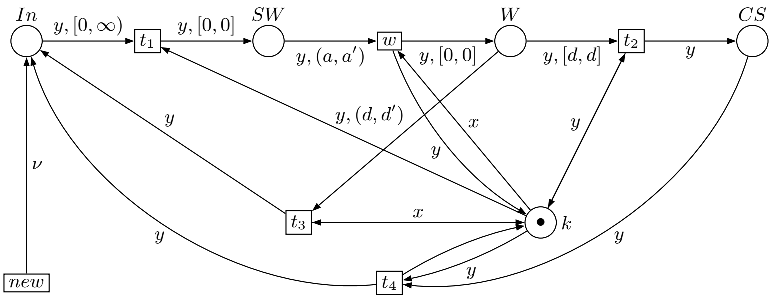

Fig. 4. Fischer's mutual exclusion protocol as a ν-*lsPN*

will be a WSTS, so that we can solve the control-state reachability problem by reducing it to a coverability problem. The proofs ommited from this section can be found in Appendix C.

We fix a ν-*lsPN* $N = \langle P, T, F, H, \mathcal{G} \rangle$ and denote by max the maximum bound appearing in the intervals of the net and $n^* \cup \{\infty\}$ as n^*_∞. Following [3, 5], we represent markings of N using *regions*.

Definition 5 (Regions). *A* region *is an expression of the form* $A_0 * A_1 * ... A_n * A_\infty$ *with* $n \geq 0$*, where* $A_i \in (P^\oplus \times I_i)^\oplus$ *for every* $i \in n^*_\infty$ *and* $I_0 = \max^*$*,* $I_i = (max - 1)^*$ *for* $i \in n^+$ *and* $I_\infty = \{\max + 1\}$*. We write* $|R| = \sum_{i \in n^*_\infty} |A_i|$*.*

We assume $A_i \neq \emptyset$ for any $i \in n^+$, and $m \neq \emptyset$ for all $(m, r) \in A_i$, for any $i \in n^*_\infty$. We use R, R',... to range over regions and $\mathcal{R}$, $\mathcal{R}'$,... to range over sets of regions. Let us intuitively explain their meaning. Each marking M of a ν-*lsPN* has a region R_M associated to it. To obtain it, we partition the instances in M into three multisets:

- The multiset M_1 of instances with an integer age of at most max,
- The multiset M_2 of instances younger than max, with a non-integer age,
- The multiset M_3 of instances older than max.

Then we put instances in M_1 in A_0, with the information about their ages (though forgetting their names). Moreover, we keep in $A_1 ... A_n$ the instances in M_2, ordered according to the fractional part of their ages, and storing only their integer part. Finally, we put instances in M_3 in A_∞, loosing the information about their age. Let us see it formally.

Definition 6 (Region of a marking). *Let* M *be a marking. We define the region* $R_M = A_0 * A^{x_1} * ... * A^{x_n} * A_\infty$ *where:*

- $A_0 = \{(m, r) \mid \exists a \in Id(M) \text{ with } m = \{p \mid p(a, r) \in M\}, ages(a, r) \in M, r \in \max^*\}$,
- $|R_M| = |Id(M)|$, $x_1, ..., x_n \in (0, 1)$ *and* $i < j$ *iff* $x_i < x_j$,
- $A^x = \{(m, \lfloor r \rfloor) \mid \exists a \in Id(M) \text{ with } m = \{p \mid p(a, r) \in M\}, ages(a, r) \in M, r < \max, \mathit{frct}(r) = x\}$,

– $A_\infty = \{(m, \max+1) \mid \exists a \in Id(M) \text{ with } m = \{p \mid p(a,r) \in M\}, ages(a,r) \in M,\ r > \max\}$.

An ν-*lsPN* N induces a transition system over regions. The definitions of time delays and firings of transitions for regions are straightforward. However, we will not give here the formal definitions, since they are rather technical. For the sake of readability, we prefer to explain the main underlying ideas instead. For the formal definitions see Appendix B.

Time elapsings: There are two ways in which time may elapse in regions. If $A_0 \neq \emptyset$, the region may evolve to $\emptyset * A_0^< * A_1 * ... * A_n * (A_\infty + A_0^=)$, where $A^< = \{(m,r) \in A \mid r < \max\}$ and $A^= = \{(m, r+1) \in A \mid r = \max\}$, which corresponds to a small elapsing of time that makes all the instances in A_0 to have a non-integer age, and so that the instances in A_n do not reach an integer age. Notice that instances in A_0 with age max are added to A_∞. If $A_0 = \emptyset$, the region may evolve to $A_n^{+1} * A_1 * ... * A_{n-1} * A_\infty$, where $A^{+1} = \{(m, r+1) \mid (m,r) \in A\}$, which represents an elapsing of time that causes the instances in A_n (those with a higher fractional part) to reach the next integer part. We write $R \overset{\delta}{\twoheadrightarrow} R'$ to denote that R' is obtained from R by elapsing of time as defined above.

Firing of transitions: A transitition over the regions corresponds to each transition t of the original net. In order to fire it, first an instance $a_j = (m_{ij}, r_{ij}) \in A_i$ is associated to each variable $x \in Var(t)$, where the corresponding preconditions hold for m_{ij} and r_{ij}. Then, the selected $a_j s$ are removed, and new pairs (m'_{ij}, r'_{ij}) are added to some place of the region according to the postconditions.

Let $\overset{\Delta}{\twoheadrightarrow}$ be the reflexive and transitive closure of $\overset{\delta}{\twoheadrightarrow}$ and $\twoheadrightarrow = \overset{\Delta}{\twoheadrightarrow} \cup \bigcup_{t \in T} \overset{t}{\twoheadrightarrow}$.

Proposition 2. *The following relations between $\rightarrow$ and $\twoheadrightarrow$ hold:*

– *If $M \longrightarrow^* M'$ then $R_M \twoheadrightarrow^* R_M$,*
– *If $R_M \twoheadrightarrow^* R'$ there is M' with $R' = R_{M'}$ and $M \longrightarrow^* M'$.*

Let us next see that we can reduce the control-state reachability problem to a coverability problem in ν-*lsPN* using regions. In the first place, we must define an order over regions, which induces the corresponding coverability problem.

Definition 7 (Order over regions). *We define $(m,r) \leq (m', r')$ iff $m \subseteq m'$ and $r = r'$. Then, we define $A_0 * A_1 * ... * A_n * A_\infty \sqsubseteq B_0 * B_1 * ... * B_m * B_\infty$ iff $A_0 \leq^\oplus B_0$, $A_\infty \leq^\oplus B_\infty$ and $A_1...A_n \leq^{\oplus\circledast} B_1...B_m$.*

Notice that we are using the word order induced by the multiset order, and therefore $\sqsubseteq$ is a decidable wpo. The order $\sqsubseteq$ induces a coverability problem in the transition system with regions as states, and we can reduce control-state reachability to it.

Proposition 3. *Given $p \in P$ we can compute a set of regions $\mathcal{R}_p$ such that p is marked by some reachable marking iff $\uparrow\mathcal{R}_p$ can be reached.*

A *Well Structured Transition System* (WSTS) is a tuple $\mathcal{S} = \langle X, \rightarrow, x_0 \leq \rangle$, where $\langle X, \rightarrow, x_0 \rangle$ is a transition system, and $\leq$ is a decidable wpo on X, such that (i) for all $x_1, x_2, x'_1 \in X$ such that $x_1 \leq x'_1$ and $x_1 \rightarrow x_2$ there is $x'_2 \in X$ such that $x'_1 \rightarrow x'_2$ and $x_2 \leq x'_2$ (compatibility); (ii) $\min(\uparrow Pre(\uparrow x))$ is computable for every $x \in X$ (effective *Pre*-basis).[4] Coverability is decidable for WSTS [1, 9].

Since $\sqsubseteq$ is a decidable wpo, in order to prove that the transition system over regions induced by a ν-*lsPNN* is a WSTS, it would only remain to prove that the transition relation is compatible with the order, and that the effective *Pre*-basis property holds. These proofs are rather technical, and therefore, we prefer to ommit them here. The proofs can be found in Appendix C. From the fact that the transition system we have defined is a WSTS, we obtain the following result:

Corollary 1. *Control-state reachability is decidable for* ν*-lsPN.*

5 Conclusions and future work

We have introduced real time in a model of dynamic networks of processes that encompasses two sources of infinity: processes can be infinite-state, and there can be infinitely-many such processes. Up to our knowledge, this is the first work in which real time is studied in this kind of concurrent systems. In the first model considered, ν-*TdPN*, each process is endowed with an arbitrary amount of real clocks, while in the second one, ν-*lsPN*, only one clock per process is allowed. While control-state reachability (whether a given place can be marked) is undecidable in the first model, we have shown that we can use the theory of regions to prove decidability of this property in the second. With regions as state space, we prove that ν-*lsPN* belong to the class of WSTS, for which coverability is decidable. In [14], we compare ν-*lsPN* with other classes of WSTS, proving that they are the most expressive of the studied classes. In particular, we prove that *TdPN* $\prec$ ν-*lsPN*. We have ommited this result from this version due to the lack of space.

As future work, we plan to study the expressive power of models in between ν-*TdPN* and ν-*lsPN*, in which a fixed number (possibly greater than one) of clocks is allowed. Also, it would be useful to compare the models yielded by bounded Petri nets with *TdPN*or NTA, in order to profit from the numerous works existing for the latter. In a different line, in our works we have assumed that processes (or their identifiers) are not ordered in any way. It would be interesting to see whether our work scales in the case of ordered processes, which amounts to extend Data Nets [13] with time.

Regarding complexity, since ν-*lsPN* are more expressive than DN or *TdPN*, the complexity of the control-state reachability problem can be proved to be non-primitive recursive. It would be interesting to obtain a finer-grained complexity analysis, as done in [11]. Although we have not discussed properties other

[4] This class is actually referred to as effective WSTS with strong compatibility and effective *Pred*-basis in the literature.

than control-state reachability, the properties of termination and boundedness are still decidable for ν-*lsPN*. Indeed, termination is decidable for WSTS under rather general hypothesis, as well as boundedness.[5] Other directions for further study include other properties, as the existence of Zeno behaviors [4], or liveness properties, although the negative results already in the untimed case are discouraging [20].

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Appendix A. Definition of ν-*RN* systems and proof of undecidability of control-state reachability

We fix an arbitrary set $\mathcal{S}$ of service names and let $Sync = \{s?, s! \mid s \in \mathcal{S}\}$.

Definition 8 (ν-*RN* systems). *A* ν-*RN* system *is a tuple* $N = \langle P, T, F, H, \lambda \rangle$, *where:*

- $\langle P, T, F, H \rangle$ *is a* ν-*PN*,
- $\lambda : T \to \mathcal{L}$ *assigns a label to each transition,*

where $\mathcal{L} = (\mathcal{S} \cup Sync) \times (P \times Id)^{\oplus}$. *An* instance *of* N *is an element of* $(P \times Id)^{\oplus}$. *A* marking *of* N *is a multiset of instances of* N.

As for ν-*TdPN*, we write $Var(t)$ to denote the set of arcs adjacent to t. For two instances M and M' we write $M \xrightarrow{t(\sigma)} M'$ if M can reach M' after the firing of t with mode σ, following the semantics of ν-*PN*. We write $Id(\mathcal{M})$ to denote the set of names that appear in marking $\mathcal{M}$. We identify any marking $\mathcal{M}$ with $\mathcal{M} + \{\emptyset\}$.[6]

Definition 9 (Firing of autonomous transitions). *Let* $t \in T$ *such that* $\lambda(t) = (s, \overline{M})$, *and* M *and* M' *be two instances such that* $M \xrightarrow{t(\sigma)} M'$ *with* $\sigma(\nu) \notin Id(\overline{M})$ *for* $\nu \in \Upsilon$. *Then* $\{M\} + \mathcal{M} \xrightarrow{t} \{M', \overline{M}\} + \mathcal{M}$ *for any marking* $\mathcal{M}$ *such that* $\sigma(\nu) \notin Id(\mathcal{M})$ *for* $\nu \in \Upsilon$.

Definition 10 (Firing of synchronizing transitions). *Let* $t_1, t_2 \in T$ *such that* $\lambda(t_1) = (s?, \overline{M}_1)$ *and* $\lambda(t_2) = (s!, \overline{M}_2)$ *for some* $s \in \mathcal{S}$, *and let* M_1, M_1', M_2 *and* M_2' *be instances such that* $M_i \xrightarrow{t_i(\sigma_i)} M_i'$ *with* $\sigma_1(x) = \sigma_2(x)$ *for all* $x \in Var(t_1) \cap Var(t_2)$ *and* $\sigma(\nu) \notin Id(\overline{M}_i)$ *for* $i = 1, 2$. *Then* $\{M_1, M_2\} + \mathcal{M} \xrightarrow{(t_1,t_2)} \{M_1', M_2', \overline{M}_1, \overline{M}_2\} + \mathcal{M}$ *for any marking* $\mathcal{M}$ *such that* $\sigma(\nu) \notin Id(\mathcal{M})$ *for* $\nu \in \Upsilon$.

Notice that when $\overline{M}$, $\overline{M}_1$ or $\overline{M}_2$ are empty then no instance is created.

Definition 11 (Control-state reachability). *We define the* control-state reachability problem *as that of deciding, given a* ν-*RN system* N *and a place* p *of* N, *whether there is a reachable marking containing an instance* M *such that* $(p, a) \in M$ *for some* $a \in Id$.

Now, we detail the proof of undecidability of control-state reachability for ν-*TdPN*. In order to further simplify the proof of the next result, we can safely assume that only autonomous transitions may create new instances (and only one in each firing), all having the same initial marking, consisting of an arbitrary token in a single place, that we can simply denote as p_0, and that initially there is a single instance. Indeed, such ν-*RN* systems are powerful enough to simulate Turing machines, as shown in [19].

[6] This is equivalent to the mechanism of $\emptyset$-expansions/contractions, though we prefer to use the later in the rest of the paper in order to deal only with wpo (and not wqo).

Proposition 4. *Control-state reachability is undecidable for ν-TdPN.*

Proof. We reduce control-state reachability for ν-*RN* systems to our problem. Given a ν-*RN* $N = \langle P, T, F, H, \lambda \rangle$, we build a ν-*TdPN* $N' = \langle P', T', F', H' \rangle$, where $P' = P \cup \{\overline{p} \mid p \in P\} \cup \{s1, s2, ins, act, q^1, ..., q^4\}$ and $T' = T \cup \{t_p \mid p \in P'\} \cup \{init, new, t_{set}\} \cup$, which simulates it. In particular, we will have $P \subset P'$, and a place $p \in P$ can be marked in N iff it can be marked in N'.

Intuitively, we represent each instance of N by a multiset of tokens with the same age in N'. The construction guarantees that all the transitions in N' use only tokens with age 1. Hence, tokens with ages older than 1 are dead tokens, that cannot be used for the firing of transitions. In order to allow instances not to become dead, we will add transitions that reset tokens with age 1 to age 0. These transitions may not reset every token with age 1, in which case some tokens are lost (after the elapsing of time).

Therefore, in some simulations some tokens are lost, but there are also perfect simulations in which no tokens are lost. In this sense our simulation is lossy, though it preserves control-state reachability, since loosing tokens can only remove behavior, so that no spurious behavior is introduced. We also guarantee in our construction that we do not merge instances, that is, that no two tokens with different ages may get to have equal ages.

Executions in N' simulate executions of N in two steps: In the first step N' creates an unbounded number of tokens with different ages, which represent all the instances that may take part in the simulation. The second step is the simulation itself. We consider in N' two places s_1 and s_2 (marked in mutual exclusion) to specify in which of the two steps the simulation is.

Step 1 (creation of instances): In the first step, depicted in Fig. 5, we repeatedly fire a transition *new*, which creates new tokens with age 0 in place *ins*. Since the age of each token in *ins* will represent a different instance of N, we need to ensure that they all have different ages. We do that by forcing some time elapsing between two consecutive firings of *new*, by demanding that the token in s_1 is strictly older than 0 when *new* is fired (and setting it to 0 when *new* is fired). Initially, there is only one token in place *ins*, with age 0.

The firing of a transition *init* concludes step 1, by moving the token in s_1 to s_2 when the token in s_1 has a non-null age. It also sets the initial marking of N, by taking a token of age 1 from *ins* and putting it in p_0, with age 0. Notice that this guarantees that the age of the token in the initial instance is different from all the ages of the tokens in *ins*.

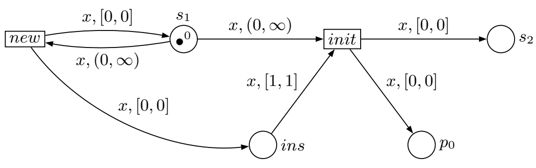


Fig. 5. Creation of instances

Step 2 (simulation of transitions): As mentioned before, only tokens with ages between 0 and 1 (both included) are valid tokens, that represent a token in some instance. Step 1 guarantees that at the beginning of step 2 there are no two tokens having ages 0 and 1. Moreover, at any point in step 2, two tokens in P with ages 0 and 1 belong to the same instance. Now we show how we reset the age of tokens, and how we simulate the firing of autonomous transitions (possibly creating a fresh instance), and the synchronization of two compatible transitions.

Reseting tokens: In order to be able to perform perfect (non-lossy) simulations, we need to be able to reset the age of tokens with age 1. For that purpose, for each place $p \in P'$ we add a transition t_p which takes from p a token of age 1 and puts it back with age 0.[7] Formally, $F'(t_p) = (p, x, [1, 1])$ and $H'(t_p) = (p, x, [0, 0])$. Notice that this is correct because before reseting there are no tokens with age 0.

Simulation of the firing of a transition: The simulation of the (autonomous) transition $t \in T$ is simply achieved by demanding that the age of all tokens involved in the firing is 1. Thus, we consider $t \in T'$, and we attach the interval $[1, 1]$ to every arc adjacent to t. More precisely, if $(p, x) \in F(t)$ then $(p, x, [1, 1]) \in F'(t)$ (and analogously for postconditions). We also add s_2 as pre/postcondition of t. Moreover, if t creates a fresh instance, it puts a token in a new place *act*. Intuitively, we store in *act* one token for each instance that the current simulation has created, but that has not been initialized yet. For the purpose of initializing new instances, we add a new transition t_{set}, which takes a token from *act* and a token with age 1 from *ins*, and puts a token in p_0 with age 0, analogously as *init* (see Fig.6). Again, notice that when there is a token with age 1 in *ins* there is no token with age 0, so that we are correctly creating the new instance.

Simulation of synchronizing transition: Let us see how we simulate the firing of $u = (t_1, t_2) \in T \times T$, where t_1 and t_2 are two compatible transitions according to λ (see Fig. 7). We simulate u by means of the consecutive firing of transitions $start_u^1$, $start_u^2$, $\overline{u}$, end_u^1 and end_u^2 in T'. We guarantee (thanks to s_2 and new control places $q^1, ..., q^4$, not shown in Fig. 7) that these transitions can only be fired in the order shown, and that $start_u^1$ can only be fired when there is a token in s_2 (no simultaneous simulations of firings can take place).

Let us consider in P' new places, $role^1$ and $role^2$ (whose content can also be reseted, as explained above), and for each $p \in P$ let us consider $\overline{p} \in P'$. The firing of $start_u^1$ removes the tokens from the preconditions p of t_1 with age 1 and puts them in the corresponding $\overline{p}$ (with any age). More precisely, if $(p, x) \in F(t_1)$ then $(p, x, [1, 1]) \in F'(start_u^1)$ and $(\overline{p}, x, [0, \infty)) \in H'(start_u^1)$. Moreover, a token (with any name, e.g. a black token) is added to $role^1$ with age 1. The case of $start_u^2$ is analogous.

The firing of $\overline{u}$ simulates the firing of u (that is, the simultaneous firing of t_1 and t_2) in the overlined places. More precisely, if $(p, x) \in F(t_i)$ for $i \in \{1, 2\}$ then $(\overline{p}, x, [0, \infty)) \in F'(\overline{u})$ (and analogously for postconditions). In particular, it checks that names in different places are matched according to the variables in

[7] It is enough to reset places in which the age is meaningful, unlike e.g. s_2.

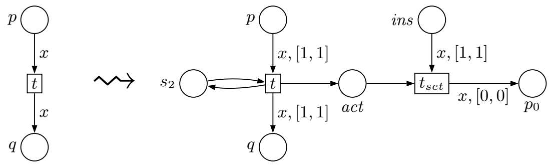


Fig. 6. Simulation of the firing of t, assuming t creates a fresh instance

the arcs, and new names are created. Notice that if the names selected by $start_u^1$ and $start_u^2$ do not match then $\overline{u}$ is disabled. Hence, our simulation may introduce deadlocks, though it still preserves control-state reachability. Notice also that this firing can take place independently of the ages of the tokens involved.

Finally, transitions end_u^1 and end_u^2 set the ages of the tokens involved in the firing of u to their correct ages. For that purpose, end_u^i takes the token from $role^i$ with age 1, and for every p postcondition of t_i it takes the token in $\overline{p}$ and puts it in p with age 1. More precisely, for $i = 1, 2$, $(role^i, y, [1, 1]) \in F'(t_i)$ (where y is a *fresh* variable), and if $(p, x) \in H(t_i)$ then $(\overline{p}, x, [0, \infty)) \in F'(end_u^i)$ and $(p, x, [1, 1]) \in H'(end_u^i)$. This concludes our simulation.

$P' = P \cup \{\overline{p} \mid p \in P\} \cup \{s1, s2, ins, act, q^1, ..., q^4\}$
$T' = T \cup \{t_p \mid p \in P'\} \cup \{init, new, t_{set}\} \cup \{start_u^i, end_u^i, \overline{u} \mid \ u = (t_1, t_2)\ compatible,\ i \in 2^+\}$

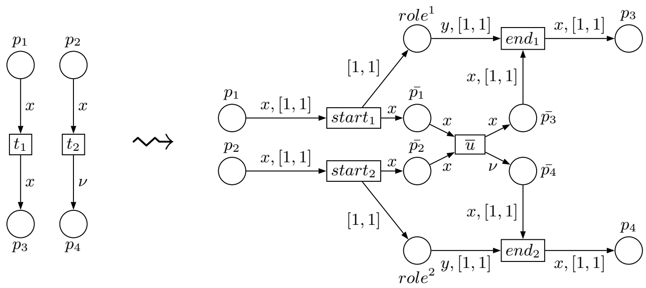


Fig. 7. Synchronizing transitions

Appendix B. Alternative definition of *ν-lsPN* and regions

Definition 12 (Locally synchronous *ν-PN*). *A locally synchronous ν-PN (ν-lsPN) is a tuple $N = \langle P, T, F, H, \mathcal{G} \rangle$, where:*

- *P and T are finite disjoint sets,*
- *for $t \in T$, $F_t, H_t : Var \to P^{\oplus}$ are the input and output functions of t,*
- *for $t \in T$, $\mathcal{G}_t : Var \to \mathcal{I} \times (\mathcal{I} \cup \{ro\})$ is the time constraints function of t.*

Definition 13 (Markings). *A* marking M *of a* ν*-lsPN is an expression of the form* $a_1{:}(m_1, r_1), ..., a_n{:}(m_n, r_n)$*, where* $Id(M) = \{a_1, ..., a_n\} \subset Id$ *are pairwise different names, and for each* $i \in n^+$*,* $\emptyset \neq m_i \in P^{\oplus}$ *and* $r_i \in \mathbb{R}_{\geq 0}$*.*

We treat markings of ν-*lsPN* as multisets over elements of the form $a{:}(m, r)$, which we call *instances*. Hence, $a : (m, r)$ is an instance with name a, tokens according to m, and age r. We assume that each m_i in each instance is not empty. We use M, M',... to range over markings. We say a marking M *marks* $p \in P$ if there is $a{:}(m, r) \in M$ such that $p \in m$.

Definition 14 (Time delay). *Given* $M = a_1 : (m_1, r_1), ..., a_n : (m_n, r_n)$ *and* $d \in \mathbb{R}_{\geq 0}$*, we write* M^{+d} *to denote the marking* $a_1{:}(m_1, r_1+d), ..., a_n{:}(m_n, r_n+d)$*, in which the age of every instance has increased by* d*. We write* $M \xrightarrow{d} M^{+d}$*.*

Now we define the firing of transitions, for which we need the following notations. We denote by $\mathcal{G}_t^1(x)$ and $\mathcal{G}_t^2(x)$ the first and second component of $\mathcal{G}_t(x)$, respectively. Intuitively, for a transition to fire the instance corresponding to x must have an age in $\mathcal{G}_t^1(x)$. This age is set to any value in $\mathcal{G}_t^2(x)$, except when x is read-only (when $\mathcal{G}_t^2(x) = ro$), in which case its age does not change. If $\nu \in \Upsilon$ we assume $\mathcal{G}_t^2(\nu) \neq ro$. For each $t \in T$ we define $Var(t) = \{x \in Var \mid F_t(x) + H_t(x) \neq \emptyset\}$, which is assumed to be finite, and we split it into $nfVar(t) = Var(t) \setminus \Upsilon$ and $fVar(t) = Var(t) \cap \Upsilon$. We say M' is an $\emptyset$*-expansion* of a marking M (or M is the $\emptyset$*-contraction* of M') if M' is obtained by adding instances $a{:}(\emptyset, r)$ to M.

Definition 15 (Firing of transitions). *Let* $t \in T$ *with* $nfVar(t) = \{x_1, ..., x_n\}$ *and* $fVar(t) = \{\nu_1, ..., \nu_k\}$*. We say* t *is* enabled *at marking* M *if:*

- $M = a_1 : (m_1, r_1), ..., a_n : (m_n, r_n) + \overline{M}$,
- *for each* $i \in n^+$*,* $F_t(x_i) \subseteq m_i$ *and* $r_i \in \mathcal{G}_t^1(x_i)$*.*

Then, t *can be* fired*, and taking*

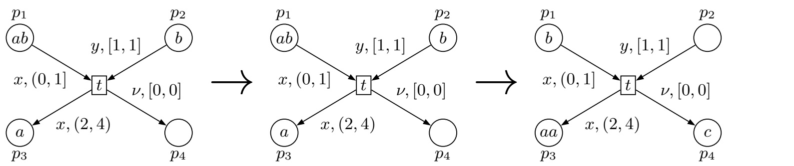
- $\{b_1, ..., b_k\}$ *pairwise different names not in* $Id(M)$*,*
- $m'_i = (m_i - F_t(x_i)) + H_t(x_i)$ *for all* $i \in n^+$*,*
- $m''_j = H_t(\nu_j)$ *for all* $j \in k^+$*,*
- $r'_i = r_i$ *if* $\mathcal{G}_t^2(x_i) = ro$*, or any value in* $\mathcal{G}_t^2(x_i)$*, otherwise, for all* $i \in n^+$*,*
- r''_j *any value in* $\mathcal{G}_t^2(\nu_j)$*, for all* $j \in k^+$*,*

we can reach M'*, denoted by* $M \xrightarrow{t} M'$*, where* M' *is the* $\emptyset$*-contraction of*

$$a_1{:}(m'_1, r'_1), ..., a_n{:}(m'_n, r'_n), b_1{:}(m''_1, r''_1), ..., b_k{:}(m''_k, r''_k) + \overline{M}$$

As for ν-*TdPN*, we write $M \to M'$ if $M \xrightarrow{t} M'$ for some $t \in T$ or $M \xrightarrow{d} M'$ for some $d \in \mathbb{R}_{\geq 0}$, and we implicitly assume an initial marking M_0, thus obtaining the transition system induced by N. Again, the semantics of ν-*lsPN* is a weak semantics, since time elapsing may disable transitions. The *control-state reachability problem* for ν-*lsPN* is defined analogously as for ν-*TdPN*, as reachability of the set of states that mark a given place.

Now, we define the regions for the new alternative definition. We denote $n^* \cup \{\infty\}$ as n^*_∞. We fix a ν-*lsPN* $N = \langle P, T, F, H, \mathcal{G} \rangle$. We simply denote by max the



$M_1 = a{:}(\{p_1p_3\}, 0),$ $b{:}(\{p_1p_2\}, 0.5)$ $\quad$ $M_2 = a{:}(\{p_1p_3\}, 0.5),$ $b{:}(\{p_1p_2\}, 1)$ $\quad$ $M_3 = a{:}(\{p_3p_3\}, 3), b{:}(\{p_1\}, 1),$ $c{:}(\{p_4\}, 0)$

Fig. 8. Firing of a transition in a ν-*lsPN*.

maximum integer bound appearing in $\mathcal{G}$. Following [3, 5], we represent markings of N using *regions*, which are expressions of the form $A_0 * A_1 * ... * A_n * A_\infty$, where each A_i represents a multiset of instances. Each marking M of a ν-*lsPN* has a region R_M associated to it. To obtain it, we partition the instances in M into three multisets:

- The multiset M_1 of instances with an integer age of at most max,
- The multiset M_2 of instances younger than max, with a non-integer age,
- The multiset M_3 of instances older than max.

Then we put instances in M_1 in A_0, with the information about their ages (forgetting their names); We keep in $A_1...A_n$ the instances in M_2, ordered according to the fractional part of their ages, and storing only their integer part; Finally, we put instances in M_3 in A_∞, forgetting about their age.

Definition 16 (Regions). *A* region *is an expression of the form* $A_0 * A_1 * ...A_n * A_\infty$ *with* $n \geq 0$*, where* $A_i \in (P^\oplus \times I_i)^\oplus$ *for every* $i \in n^*_\infty$ *and* $I_0 = \max^*$, $I_i = (max - 1)^*$ *for* $i \in n^+$ *and* $I_\infty = \{\max + 1\}$*. We write* $|R| = \sum_{i \in n^*_\infty} |A_i|$.

We assume $A_i \neq \emptyset$ for any $i \in n^+$, and $m \neq \emptyset$ for all $(m, r) \in A_i$, for any $i \in n^*_\infty$. We use R, R',... to range over regions and $\mathcal{R}$, $\mathcal{R}'$,... over sets of regions.

Definition 17 (Region of a marking). *Let* M *be a marking. We define the region* $R_M = A_0 * A^{x_1} * ... * A^{x_n} * A_\infty$ *where:*

- $|R_M| = |M|$, $x_1, ..., x_n \in (0, 1)$ *and* $i < j$ *iff* $x_i < x_j$,
- $A_0 = \{(m, r) \mid a{:}(m, r) \in M,\ r \in \max^*\}$,
- $A^x = \{(m, \lfloor r \rfloor) \mid a{:}(m, r) \in M, r < \max, \mathit{frct}(r) = x\}$,
- $A_\infty = \{(m, \max + 1) \mid a{:}(m, r) \in M,\ r > \max\}$.

Example 2. Let $M = a_1{:}(\{p\}, 1), a_2{:}(\{p\}, 1.1), a_3{:}(\{q\}, 2.1), a_4{:}(\{p, q\}, 1.2), a_5{:}(\{pq\}, 3.1)$, and $\max = 3$. Then, $R_M = A_0 * A_1 * A_2 * A_\infty$, with $A_0 = \{(\{p\}, 1)\}$ (which represents the only instance with integer age), $A_1 = \{(\{p\}, 1), (\{q\}, 2)\}$, $A_2 = \{(\{p, q\}, 1)\}$ (corresponding to the two different fractional parts, ordered) and $A_\infty = \{(\{pq\}, 4)\}$ (the only instance with age greater than max).

We now define the transition system over regions induced by N. For the next definition we introduce the following notations: For $A \in (P^\oplus \times I)^\oplus$ let $A^< = \{(m, r) \in A \mid r < \max\}$, $A^= = \{(m, r + 1) \in A \mid r = \max\}$ and $A^{+1} = \{(m, r + 1) \mid (m, r) \in A\}$. Moreover, if $(m, 0) \notin A$ for any $m \in P^\oplus$ then A^{-1} is defined as $\{(m, r - 1) \mid (m, r) \in A\}$.

Definition 18 (Time elapsing for regions). *Let* $R = A_0 * A_1 * \ldots * A_n * A_\infty$ *be a region. We write* $R \overset{\delta}{\twoheadrightarrow} R'$ *if:*

$$R' = \begin{cases} \emptyset * A_0^< * A_1 * ... * A_n * (A_\infty + A_0^=) & \text{if } A_0 \neq \emptyset \\ A_n^{+1} * A_1 * ... * A_{n-1} * A_\infty & \text{otherwise} \end{cases}$$

When $A_0 \neq \emptyset$ a δ step corresponds to a small elapsing of time that makes all the instances in A_0 have a non-integer age, and so that the ones in A_n do not reach an integer age. Note that instances in A_0 with age max are added to A_∞. The case $A_0 = \emptyset$ represents an elapsing of time that causes the instances in A_n (with the highest fractional part) to reach the next integer. In order to define the firing of transitions for regions we need $\emptyset$-expansions/contractions for them.

Definition 19 ($\emptyset$-expansion/contraction). *We say* R' *is an* $\emptyset$-expansion *of a region* $R = A_0 * A_1 * ... * A_n * A_\infty$ *(or* R *is the* $\emptyset$-contraction *of* R'*) if* R' *is of the form* $A'_0 * u_0 * A'_1 * u_1 * ... * A'_n * u_n * A'_\infty$ *and for each* i*:*

- $A'_i = A_i + B_i$ *with* $m = \emptyset$ *for all* $(m, r) \in B_i$*,*
- $u_i = B^i_1 * ... * B^i_{k_i}$ *with* $k_i \geq 0$ *and* $m = \emptyset$ *for all* $(m, r) \in B^i_j$*.*

Finally, we define how regions evolve when a transition is fired. Given a region $R = A_0 * A_1 * ... * A_n * A_\infty$, we will write variables as x_{ij}, meaning that x_{ij} is instantiated to the j-th instance in A_i. Then, we will use a function τ, so that $\tau(i, j)$ is the position in the resulting region in which that instance is put. If $\tau(i, j) \in n^*_\infty$ then it is sent to one of the positions already existing in R. Otherwise, it is of the form (i', k), meaning that it is put in a new position between i-th and the $(i + 1)$-th ones.

For any interval I, we call *left closure of* I the result of replacing the left delimiter of I by a closed one (for instance, the left closure of (a, b) is $[a, b)$).

Now we define the firings of transitions for regions. Intuitively, a transition t is enabled at a region if we can assign each variable $x \in Var(t)$ with $x \notin \Upsilon$ to a pair (m, r) in some A_i of the region, in such a way that $F_t(x) \subseteq m$ and the age that represents the pair is in $\mathcal{G}^1_t(x)$. Then, the transition can be fired, reaching a new region in which we update the markings of the pairs assigned to each variable according to F_t and H_t, and we update the ages of the pair according to $\mathcal{G}^2_t$. Moreover, we possibly need to remove some of the pairs we have choosen from the A_is they are in, and set them into other A_js, according one of the possible ages they may represent. More precisely, the only pairs we need to keep in the same A_is are those assigned to variables with $\mathcal{G}^2_t(x) = ro$. Finally, for each $\nu \in \Upsilon$, we set a new pair $(H_t(\nu), r)$ in a proper (and maybe new) multiset A of the region. The formal definition of firings of transitions is the following one:

Definition 20 (Firing of transitions for regions). *Let* R *be a region of* N*,* $t \in T$ *and let* $l = |Var(t)|$*. We say* t *is enabled in* R *if there is an* $\emptyset$-*expansion* $A_0 * A_1 * ... * A_n * A_\infty$ *of* R*,* $Var(t) = \{x_{ij} \mid i \in n^*_\infty, j \in k^+_i\}$ *and* $\bar{A}_i = \{(m_{ij}, r_{ij}) \mid j \in k^+_i\} \subseteq A_i$ *for some* $k_0, k_1, ...k_n, k_\infty \geq 0$*, such that:*

– *If* $x_{ij} \in \Upsilon$ *then* $m_{ij} = \emptyset$.
– $F_t(x_{ij}) \subseteq m_{ij}$.
– $r_{ij} \in \mathcal{G}^1_t(x_{ij})$ *if* $i \in \{0, \infty\}$, *and* $r_{ij} + 0.5 \in \mathcal{G}^1_t(x_{ij})$, *otherwise.*

Then, we take:

– $B_i = A_i - \bar{A}_i$.
– $m'_{ij} = (m_{ij} - F_t(x_{ij})) + H_t(x_{ij})$.
– $r'_{ij} = r_{ij}$ *if* $\mathcal{G}^2_t(x_{ij}) = ro$, *and* $r'_{ij} \in (\max + 1)^*$ *in the left closure of* $\mathcal{G}^2_t(x_{ij})$, *otherwise.*
– $\tau(i,j) \in n^*_\infty \cup (n^*_\infty \times l^+)$ *such that:*
  - *if* $\mathcal{G}^2_t(x_{ij}) = \mathcal{G}^2_t(x_{ks}) = ro$, $i < k$ *iff* $\tau(i,j)_1 < \tau(k,s)_1$, *where* $\tau(i,j)_1 = i'$ *if* $\tau(i,j) = i'$ *or* $\tau(i,j) = (i', j')$ *for some* j';
  - *if* $r'_{ij} > \max$ *then* $\tau(i,j) = \infty$;
  - *if* $\mathcal{G}^2_t(x_{ij}) = (a, b]$ *or* $\mathcal{G}^2_t(x_{ij}) = (a, b)$ *and* $r'_{ij} = a$ *then* $\tau(i,j) \neq 0$.

Finally, let us define $D_k = \{(m'_{ij}, r'_{ij}) \mid \tau(i,j) = k\}$, $C_k = B_k + D_k$ *and* $C_{ab} = \{(m'_{ij}, r'_{ij}) \mid \tau(i,j) = (a,b)\}$. *Then, we write* $R \xrightarrow{t}\!\!\!\!\rightarrow R'$, *where* R' *is the* $\emptyset$-*contraction of* $C_0 * C_{01} * \ldots * C_{0l} * C_1 * C_{11} * \ldots * C_{1l} * \ldots * C_n * C_{n1} * \ldots * C_{nl} * C_\infty$.

Appendix C. Proofs of Section 4.

C1. Proof of Proposition 3

Let us denote $\mathcal{C}(M) = \{r \mid a{:}(m,r) \in M\} \in \mathbb{R}_{\geq 0}{}^{\oplus}$.

Lemma 1. *Let* M *be a marking such that* $\mathcal{C}(M) \cap \mathbb{N} \neq \emptyset$ *and* $\epsilon = \max\{frct(r) \mid r \in \mathcal{C}(M)\}$. *If* $0 < d < 1 - \epsilon$ *then* $R_M \xrightarrow{\delta}\!\!\!\!\rightarrow R_{M^{+d}}$. *Moreover,* $\mathcal{C}(M^{+d}) \cap \mathbb{N} = \emptyset$.

Proof. Suppose that $R_M = A_0 * A^{x_1} * ... * A^{x_n} * A_\infty$, where $0 < x_1 < ... < x_n < 1$ are the fractional parts of the ages of the instances younger than max in M. Then, as $x_n = \epsilon = \max\{frct(r) \mid r \in \mathcal{C}(M)\}$, $0 < d < 1 - \epsilon$ and $\mathcal{C}(M) \cap \mathbb{N} \neq \emptyset$, the fractional parts of the ages of the instances younger than max in M^{+d} are $d, x_1 + d, ..., x_n + d$ (with $x_n + d < 1$). For each $i \in n^+$, the instances and markings with fractional parts of its ages $x_i + d$ in M^{+d} are the same as the ones in M with fractional parts of its ages x_i. Moreover, the instances and markings with fractional parts of its ages d in M^{+d} are the instances with natural ages younger than max in M. Therefore, $R_{M^{+d}} = \emptyset * A^d * A^{x_1+d} * ... * A^{x_n+d} * A'_\infty$ as defined in Def. 17, where in A'_∞ are represented the instances in A_∞ and the instances in A_0 with age max and in A^d are represented the instances in A_0 younger than max. By the first case of Def. 18, we have that $R_M \xrightarrow{\delta}\!\!\!\!\rightarrow R'$, where $R' = \emptyset * A_0^< * A_{x_1} * ... * A_{x_n} * (A_\infty + A_0^=) = \emptyset * A^d * A^{x_1+d} * ... * A^{x_n+d} * A'_\infty = R_{M^{+d}}$.

Lemma 2. *Let* M *be a marking such that* $\mathcal{C}(M) \cap \mathbb{N} = \emptyset$ *and* $\epsilon = \max\{frct(r) \mid r \in \mathcal{C}(M)\}$. *If* $d < 1 - \epsilon$ *then* $R_M = R_{M^{+d}}$.

Proof. Suppose that $R_M = A_0 * A^{x_1} * ... * A^{x_n} * A_\infty$, where $0 < x_1 < ... < x_n < 1$ are the fractional parts of the ages of the instances younger than max in M. Then, as $x_n = \epsilon = \max\{frct(r) \mid r \in \mathcal{C}(M)\}$, $d < 1 - \epsilon$ and $\mathcal{C}(M) \cap \mathbb{N} = \emptyset$, the fractional parts of the ages of the instances younger than max in M^{+d} are $x_1 + d, ..., x_n + d < 1$, and moreover, for each $i \in n^+$, the instances and markings with fractional parts of its ages $x_i + d$ in M^{+d} are the same as the ones in M with fractional parts of its ages x_i. Therefore, by the definition of region, $R_{M^{+d}} = A_0 * A^{x_1+d} * ... * A^{x_n+d} * A_\infty = A_0 * A^{x_1} * ... * A^{x_n} * A_\infty = R_M$.

Lemma 3. *Let M be a marking such that $\mathcal{C}(M) \cap \mathbb{N} = \emptyset$ and $\epsilon = \max\{frct(r) \mid r \in \mathcal{C}(M)\}$. If $d = 1 - \epsilon$ then $R_M \overset{\delta}{\twoheadrightarrow} R_{M^{+d}}$. Moreover, $\mathcal{C}(M^{+d}) \cap \mathbb{N} \neq \emptyset$.*

Proof. Suppose that $R_M = \emptyset * A^{x_1} * ... * A^{x_n} * A_\infty$, where $x_1, ..., x_n \in (0, 1)$ with $i < j$ iff $x_i < x_j$, are the fractional parts of the ages of the instances younger than max in M. Then, as $x_n = \epsilon = \max\{frct(r) \mid r \in \mathcal{C}(M)\}$, $d = 1 - \epsilon$ and $\mathcal{C}(M) \cap \mathbb{N} = \emptyset$, the fractional parts of the ages of the instances younger than max in M^{+d} are $0, x_1 + d, ..., x_{n-1}$. For each $i \in n - 1^+$, the instances and markings with fractional parts of its ages $x_i + d$ in M^{+d} are the same as the ones in M with fractional parts of its ages x_i. Moreover, the instances and markings with natural ages in M^{+d} are the instances with ages with fractional part x_n in M. Therefore, $R_{M^{+d}} = A_0 * A^{x_1+d} * ... * A^{x_{n-1}+d} * A_\infty$ as defined in Def. 17. By the second case of Def. 18, we have that $R_M \overset{\delta}{\twoheadrightarrow} R'$, where $R' = A_n^{+1} * A_{x_1} * ... * A_{x_{n-1}} * A_\infty = A_0 * A^{x_1+d} * ... * A^{x_{n-1}+d} * A_\infty = R_{M^{+d}}$.

Lemma 4. *Let M be a marking such that $\mathcal{C}(M) \cap \mathbb{N} \neq \emptyset$. Then $R_M \overset{\Delta}{\twoheadrightarrow} R_{M^{+1}}$.*

Proof. Let $R_M = A_0 * A_1 * ... * A_n * A_\infty$ and let x_i be the fractional part of the ages of instances in A_i for $i \in n^*$ (so $x_0 = 0$) and take $x_{n+1} = 1$. Then $x_0 < x_1 < ... < x_n < x_{n+1}$. We define $\epsilon_i = (x_{i+1} - x_i)/2$ for $i \in n^*$. Let $M_{n+1} = M$, $M'_{i+1} = M_{i+1}^{+\epsilon_i}$ for $i \in n^*$ and $M_{i-1} = (M'_i)^{+\epsilon_i}$ for $i \in (n + 1)^+$. Then we have $M = M_{n+1} \overset{\epsilon_n}{\twoheadrightarrow} M'_{n+1} \overset{\epsilon_n}{\twoheadrightarrow} M_n \overset{\epsilon_{n-1}}{\twoheadrightarrow} ... \overset{\epsilon_1}{\twoheadrightarrow} M_1 \overset{\epsilon_0}{\twoheadrightarrow} M'_1 \overset{\epsilon_0}{\twoheadrightarrow} M_0$. Notice that $\sum_{i \in n^*} 2\epsilon_i = 1$, so that $M_0 = M^{+1}$. It also holds that $\mathcal{C}(M_i) \cap \mathbb{N} \neq \emptyset$ for all $i \in (n + 1)^*$ and $\mathcal{C}(M'_i) \cap \mathbb{N} = \emptyset$ for all $i \in (n + 1)^+$. Moreover, the maximum fractional part of the reals in M_{i+1} is $1 - 2\epsilon_i$ for $i \in n^+$, and that of M'_{i+1} is $1 - \epsilon_i$ for $i \in n^*$. Then M_{i+1} and ϵ_i are in the hypothesis of Lemma 1, and M'_{i+1} and ϵ_i in the ones of Lemma 3. Therefore, $R_{M_i} \overset{\delta}{\twoheadrightarrow} R_{M'_i}$ for $i \in (n + 1)^+$ and $R_{M'_{i+1}} \overset{\delta}{\twoheadrightarrow} R_{M_i}$ for $i \in n^*$, so that $R_M = R_{M_{n+1}} \overset{\Delta}{\twoheadrightarrow} R_{M_0} = R_{M^{+1}}$.

Proposition 5. *We have the following:*

1. *If $M \overset{t}{\longrightarrow} M'$ then $R_M \overset{t}{\twoheadrightarrow} R'_M$,*
2. *If $M \overset{d}{\longrightarrow} M'$ then $R_M \overset{\Delta}{\twoheadrightarrow} R_{M'}$,*
3. *If $R_M \overset{t}{\twoheadrightarrow} R'$ there is M' with $R' = R_{M'}$ and $M \overset{t}{\longrightarrow} M'$.*
4. *If $R_M \overset{\delta}{\twoheadrightarrow} R'$ there is M' with $R' = R_{M'}$ and $M \overset{d}{\longrightarrow} M'$ for some $d \in (0, 1)$,*

5. *If* $R_M \overset{\Delta}{\twoheadrightarrow} R'$ *there is* M' *with* $R' = R_{M'}$ *and* $M \overset{d}{\longrightarrow} M'$ *for some* $d \in \mathbb{R}_{\geq 0}$.

Proof.

(1) Let us suppose that $M \overset{t}{\longrightarrow} M'$. Then, if $nfVar(t) = \{x_1, ..., x_{n_1}\}$ and $fVar(t) = \{x_{n_1+1}, ..., x_{n_2}\}$ then we have $M = a_1 : (m_1, r_1), ..., a_{n_1} : (m_{n_1}, r_{n_1}) + \overline{M}$ and for each $i \in n_1^+$, $F_t(x_i) \subseteq m_i$ and $r_i \in \mathcal{G}_t^1(x_i)$. Moreover, let $M'' = a_1 : (m''_1, r''_1), ..., a_{n_1} : (m''_{n_1}, r''_{n_1}), a_{n_1+1} : (m''_{n_1+1}, r''_{n_1+1}), ..., a_{n_2} : (m''_{n_2}, r''_{n_2}) + \overline{M}$ be the $\emptyset$-expansion of M' obtained in the process of firing t. Let $A_0 * A_1 * \ldots * A_n * A_\infty$, with $A_i = \{(m_{i1}, r_{i1}), \ldots, (m_{ik_i}, r_{ik_i})\}$ be an $\emptyset$-expansion of R_M and $R_{M''} = A''_0 * A''_1 * \ldots * A''_{n''} * A''_\infty$, with $A''_i = \{(m''_{i1''}, r''_{i1''}), \ldots, (m''_{ik''_i}, r''_{ik''_i})\}$ be the region of M'', with $l = max\{k''_i \mid i \in n''^{*}_\infty\}$. Let $\Phi : n_2^+ \to n_\infty^* \times l$ be the function which associates each i to the location of the pair which represents the instance a_i in R_M, if $i < n_1$, and the location of a pair with empty marking, otherwise. Let us rename $Var(t) = \{x_{\Phi(i)} \mid i \in n_2^+\}$. Then, as t is enabled in M, we have that:

- If $x_{ij} \in \Upsilon$ then $m_{ij} = \emptyset$.
- $F_t(x_{ij}) \subseteq m_{ij}$ because of the enabling condition of t.
- $r_{ij} \in \mathcal{G}_t^1(x_{ij})$ if $i \in \{0, \infty\}$, and $r_{ij} + 0.5 \in \mathcal{G}_t^1(x_{ij})$, otherwise.

Therefore, we can fire t from R_M. Now, we prove that we can fire it in such a way that we reach $R_{M''}$. For that purpose, we first need to define the proper function τ for the firing.

Suppose that $\Phi(g) = (i, j)$. Then, we define:

- $\tau(i, j) = 0$ if $frct(r''_g) = 0$.
- $\tau(i, j) = h$ if $frct(r''_g)$ is the fractional part which has names represented by A_h in R_M and $r''_g < max$.
- $\tau(i, j) = \infty$ if $r''_g > max$.
- $\tau(i, j) = (h_1, h_2)$ if A_{h_1} represents the names with the greatest fractional part f lower than $frct(r''_g)$, and $frct(r''_g)$ is the $h_2 - th$ fractional part greater than f.

It is easy to see that the three conditions of the definition of τ hold. Indeed:

1. Suppose $x_{ij}, x_{ks} \in Var(t)$ with $\mathcal{G}_t^2(x_{ij}) = \mathcal{G}_t^2(x_{ks}) = ro$ and $\phi(g_1) = (i, j)$ and $\phi(g_2) = (k, s)$. Then, if $k \neq \infty$, $i < k \Leftrightarrow frct(r_{g_1}) = frct(r''_{g_1}) < frct(r''_{g_2}) = frct(r_{g_1})$ and $r_{g_1} = r''_{g_1} < max, r_{g_2} = r_{g_2} < max \Leftrightarrow \tau(i, j)_1 = i < k = \tau(k, s)_1$. If $k = \infty$, $i < k \Leftrightarrow r_{g_1} = r''_{g_1} < max, frct(r_{g_1}) = frct(r''_{g_1})$ and $r_{g_2} = r_{g_2} > max \Leftrightarrow \tau(i, j)_1 = i < \infty = k = \tau(k, s)_1$.
2. Suppose $x_{i,j} \in Var(t)$, $\Phi(g) = (i, j)$ with $r''_g > max$. Then, $\tau(i, j) = \infty$.
3. Finally, suppose $x_{i,j} \in Var(t)$, $\Phi(g) = (i, j)$ with $\mathcal{G}_t^2(x_{ij}) = (a, b]$ or $\mathcal{G}_t^2(x_{ij}) = (a, b)$. Then, if $\lfloor r''_{ij} \rfloor = a$ then $frct(r''_{ij}) > 0$ and therefore $\tau(i, j) \neq 0$.

Finally, let us prove that the region $R = C_0 * C_{01} * \ldots * C_{0l} * C_1 * C_{11} * \ldots * C_{1l} * \ldots * C_n * C_{n1} * \ldots * C_{nl} * C_\infty$ reached after firing t from R_M with τ, the already defined renaming of $Var(t)$ and choosing $\lfloor r''_k \rfloor$ as r'_{ij} of Def. 15 if $\Phi(k) = (i, j)$; is $R_{M''}$, except, maybe, for the empty markings and empty $A''_i s$.

We are going to prove that A''_i is in $R_{M''}$ iff it is in R.

First, suppose that the fractional part of the age of the instances represented by A''_i is x, and this instances are younger than or equal to max. We are going to analyse two different cases:

- Suppose that there are instances with fractional part x in M, represented by A^x. We know that $A''_i = \{(m, \lfloor r \rfloor) \mid a{:}(m,r) \in M,\ r < \max,\ \mathit{frct}(r) = x\}$. Then, according to this definition and the definition of firings we have: $A''_i = (\{(m, \lfloor r \rfloor) \mid \exists a \text{ with } a : (m,r) \in \bar{M},\ r \leq \max,\ \mathit{frct}(r) = x\}) + (\{(m, \lfloor r \rfloor) \mid \exists b \text{ a free name with } b : (m,r) \in M'', m = H_t(\nu) \text{ for some } \nu \in fVar(t) \text{ and } r \in \mathcal{G}^2_t(\nu),\ r \leq \max,\ \mathit{frct}(r) = x\} + \{(m, \lfloor r \rfloor) \mid \exists a_j, m_j, r_j \text{ with } a_j{:} (m_j, r_j) \in M, m = (m_j - F_t(x_j)) + H_t(x_j) \text{ and } r = r_j \text{ if } \mathcal{G}^2_t(x_j) = ro$ $\text{, or any value in } \mathcal{G}^2_t(x_i) \text{ ow.},\ r \leq \max,\ \mathit{frct}(r) = x\}) = (A^x - \bar{A^x}) + (D_h) = C_h$, where $\bar{A^x}$ and D_h are as in the Def. 20, and h such that $A^x = A_h$.
- Now, suppose that there are not instances with fractional part x represented in M and $A''_i \neq \emptyset$. Then, we have: $A''_i = \{(m, \lfloor r \rfloor) \mid \exists b \text{ a free name with } b{:}(m,r) \in M'', m = H_t(\nu) \text{ for some } \nu \in fVar(t) \text{ and } r \in \mathcal{G}^2_t(\nu),\ r < \max,\ \mathit{frct}(r) = x\} + \{(m, \lfloor r \rfloor) \mid \exists a_j, m_j, r_j \text{ with } a_j{:} (m_j, r_j) \in M, m = (m_j - F_t(x_j)) + H_t(x_j) \text{ and } r \text{ is any value in } \mathcal{G}^2_t(x_i),\ r < \max,\ \mathit{frct}(r) = x\} = \{(m''_{h_3,h_4}, r''_{h_3,h_4}) \mid \tau(h_3, h_4) = (h_1, h_2)\} = C_{h_1 h_2}$, where $C_{h_1 h_2}$ is as in the Def. 20 and A_{h_1} represents the names with the greatest fractional part f lower than $\mathit{frct}(x)$, and $\mathit{frct}(x)$ is the $h_2 - th$ fractional part greater than f.

Then, the only case left is A''_∞:
$A''_\infty = (\{(m, \lfloor r \rfloor) \mid \exists a \text{ with } a : (m,r) \in \bar{M},\ r > \max\}) + (\{(m, \lfloor r \rfloor) \mid \exists b \text{ a free name with } b{:}(m,r) \in M'', m = H_t(\nu) \text{ for some } \nu \in fVar(t) \text{ and } r \in \mathcal{G}^2_t(\nu),\ r > \max\} + \{(m, \lfloor r \rfloor) \mid \exists a_j, m_j, r_j \text{ with } a_j : (m_j, r_j) \in M, m = (m_j - F_t(x_j)) + H_t(x_j) \text{ and } r = r_j \text{ if } \mathcal{G}^2_t(x_j) = ro, \text{ or any value in } \mathcal{G}^2_t(x_i) \text{ ow.},\ r > \max\}) = (A_\infty - \bar{A_\infty}) + (D_\infty) = C_\infty$, where $\bar{A_\infty}$ and D_∞ are as in Def. 20.

Finally, note that the order of the obtained $C_i s$ of R correspond to the order of the corresponding A''_i of $R_{M''}$. That is because we have defined τ, in such a way that we order the different $C_i s$ depending on the fractional part of $r''_g s$ younger than max, setting the instances older than max in C_∞, as in $R_{M''}$.

(2) Let $d > 1$ (the other case is easier) and $M \xrightarrow{d} M'$. Suppose that $\mathcal{C}(M) \cap \mathbb{N} \neq \emptyset$ (otherwise, by lemma 3 we know that there exist ϵ such that $R_M \overset{\delta}{\twoheadrightarrow} R_{M^{+\epsilon}}$ and $\mathcal{C}(M^{+\epsilon}) \cap \mathbb{N} \neq \emptyset$, and we start from $M^{+\epsilon}$). Then, we have that $M \xrightarrow{1} M^{+1} \xrightarrow{1} M^{+2} \xrightarrow{1} \ldots \xrightarrow{1} M^{+\lfloor d \rfloor} \xrightarrow{\mathit{frct}(d)} M'$. Because of Lemma 4 we know that $R_M \overset{\Delta}{\twoheadrightarrow} R_{M^{+1}} \overset{\Delta}{\twoheadrightarrow} R_{M^{+2}} \overset{\Delta}{\twoheadrightarrow} \ldots \overset{\Delta}{\twoheadrightarrow} R_{M^{+\lfloor r \rfloor}}$. Therefore, we only need to prove that $R_{M^{+\lfloor d \rfloor}} \overset{\Delta}{\twoheadrightarrow} R_{M'}$. As in Lemma 4, let $R_{M^{+\lfloor d \rfloor}} = A_0 * A_1 * ... * A_n * A_\infty$ and let x_i be the fractional part of the ages of instances in A_i for $i \in n^*$, $x_{n+1} = 1$, $\epsilon_i = (x_{i+1} - x_i)/2$ for $i \in n^*$, $M_{n+1} = M^{+\lfloor d \rfloor}$, $M'_{i+1} = M^{+\epsilon_i}_{i+1}$ and $M_{i-1} = (M'_i)^{+\epsilon_i}$. Now, we select k such that $1 - x_k \leq \mathit{frct}(d)$ and $1 - x_{k-1} > \mathit{frct}(d)$, and we define $y = x_k - (1 - \mathit{frct}(d))$ and $M_y = M^{+y}_k$. Note that $\sum_{i=k}^{n} 2 * \epsilon + y = (1 - x_k) + x_k - (1 - \mathit{frct}(d)) = \mathit{frct}(d)$, so that $M_y = {M^{+\lfloor d \rfloor}}^{+\mathit{frct}(d)} = M'$. Repeating the same reasoning as in Lemma 4, we can conclude that $R_{M^{+\lfloor d \rfloor}} = R_{M_{n+1}} \overset{\Delta}{\twoheadrightarrow} R_{M_y} = R_{M'}$.

(3) Suppose that $R_M \stackrel{t}{\twoheadrightarrow} R'$, with τ. Then, we know that there is an $\emptyset$-expansion $A_0 * A_1 * ... * A_n * A_\infty$ of R_M, $Var(t) = \{x_{ij} \mid i \in n^*_\infty, j \in k_i^+\}$ and $\bar{A}_i = \{(m_{ij}, r_{ij}) \mid j \in k_i^+\} \subseteq A_i$ such that:

- $F_t(x_{ij}) \subseteq m_{ij}$.
- $r_{ij} \in \mathcal{G}^1_t(x_{ij})$ if $i \in \{0, \infty\}$, and $r_{ij} + 0.5 \in \mathcal{G}^1_t(x_{ij})$, otherwise.

Therefore, we know that $M = M_s + \bar{M}$, where $M_s = \{a_{ij}{:}(m_{ij}, r'_{ij})) \mid (m_{ij}, r_{ij}) \in A_i\}$, where $r_{ij} = \lfloor r'_{ij} \rfloor$, such that, if $(m_{ij}, r'_{ij}) \in M_s$ then $F_t(x_{ij}) \subseteq m_{ij}$ and $r'_{ij} \in \mathcal{G}^1_t(x_{ij})$, and therefore, t is enabled at M.

Suppose $R' = A'_0 * A'_1 * ... * A'_{n'} * A'_\infty$. Now, in order to define the marking M', for each $i \in n'^+$ we are going to define $d_i \in [0, 1)$ such that the fractional part of the age of instances represented in A'_i is d_i. Given i, if A'_i comes from a C_j in the definition of firings of transitions for regions such that $A_j - \bar{A}_j \neq \emptyset$ we define d_i as the fractional part of the age of the instances represented by A_j in M. Otherwise, we define d_i as any number which keeps the d_js ordered by the natural order. Moreover, we define $d_\infty = d_0 = 0$.

Then, we write $R' = A'_0 * A^{d_1} * ... * A^{d_n} * A'_\infty$. Now, let us consider the marking M' of the following form:

- If $a{:}(m, r) \in \bar{M}$ then $a{:}(m, r) \in M'$.
- If $a_{ij}:(m_{ij}, r) \in M$, such that $a_{ij}:(m_{ij}, r'_{ij}) \in A_i$ represents it in R_M, $\tau(i, j) = k$ and $x_{ij} \in Var(t)$, then $a_{ij}{:}(m', r') \in M'$, where:
  - If $x_{ij} \in nfVar(t)$ then $m' = (m_{ij} - F_t(x_{ij})) + H_t(x_{ij})$ and if $x_{ij} \in fVar(t)$ then $m' = m_{ij} + H_t(x_{ij})$
  - $r' = r'_{ij} + d_k$, where r'_{ij} is defined as in the firing of transitions t for R_M.

  Let us call M_{nf} the set of $a_{ij}:(m_{ij}, r) \in M$ corresponding to non-free variables, and M_f the set $a_{ij}{:}(m_{ij}, r) \in M$ corresponding to free variables.

Then, $M \to M'$ because $M' = M_s + M_{nf} + M_f$ and for each $x_{ij} \in nfVar(t)$ there is $a_{ij}{:}(m, r) \in M$ and $a_{ij}{:}(m_{ij}, r') \in M_{nf}$ such that $m_{ij} - F_t(x_{ij}) + H_t(x_{ij})$, $r' = r$ if $\mathcal{G}^2_t(x_{ij}) = ro$, or $r' \in \mathcal{G}^2_t(x_i)$, and anagously, for each $x_{ij} \in fVar(t)$ there is $a_{ij}{:}(m_{ij}, r') \in M_f$ such that $m_{ij} = H_t(x_{ij})$ and $r' \in \mathcal{G}^2_t(x_i)$.

M' and each d_i, it is easy to see that $R' = R_{M'}$.

(4) Let us suppose that $R_M \stackrel{\delta}{\twoheadrightarrow} R'$. Let us first consider the case in which $R_M = \emptyset * A^{x_1} * ... * A^{x_n} * A_\infty$. Then, $R' = A_n^{+1} * A^{x_1} * ... * A^{x_n} * A_\infty$. Let d be the fractional part of the ages of the instances in A_n. And let M' be a marking such that $M \xrightarrow{d} M'$. We are going to prove that $R' = R_{M'}$. By the definition of region of a marking, $R_{M'} = A'_0 * A^{x'_1} * ... * A^{x'_n} * A'_\infty$ where:

- $A'_0 = \{(m, r) \mid a{:}(m, r) \in M',\ r \in \max^*\} = \{(m, r) \mid a{:}(m, r - (1 - d)) \in M,\ r \in \max^*\} = A_n^{+1}$.
- $A^{x'_i} = \{(m, \lfloor r \rfloor) \mid a{:}(m, r) \in M', r < \max, frct(r) = x'_i\} = \{(m, \lfloor r \rfloor) \mid a{:}(m, r - d) \in M, r - d < \max, frct(r - d) = x_i\} = A^{x_i}$.
- $A_\infty = \{(m, \max + 1) \mid a{:}(m, r) \in M',\ r > \max\} = \{(m, \max + 1) \mid a{:}(m, r - d) \in M,\ r - d > \max\} = A_\infty$.

Now, we consider the case in which $R_M = A_0 * A^{x_1} * ... * A^{x_n} * A_\infty$ and $A_0 \neq \emptyset$. Then, $R' = \emptyset * A_0^{<} * A^{x_1} * ... * A^{x_n} * (A_\infty + A_0^{=})$. Let $0 < d < 1 - x_n$, and let M' be a marking such that $M \xrightarrow{d} M'$. Again, we are going to prove that $R' = R_{M'}$. By the definition of region of a marking, $R_{M'} = A'_0 * A^{x'_1} * ... * A^{x'_n} * A'_\infty$ where:

- $A'_0 = \{(m,r) \mid a{:}(m,r) \in M',\ r \in \max^*\} = \{(m,r) \mid a{:}(m,r-d) \in M',\ r \in \max^*\} = \emptyset$.
- $A^{x'_1} = \{(m,\lfloor r \rfloor) \mid a{:}(m,r) \in M', r < \max, \mathit{frct}(r) = x'_1\} = \{(m,\lfloor r \rfloor) \mid a{:}(m,r-d) \in M, r < \max, \mathit{frct}(r) = x'_1\} = A_0^{<}$.
- $A^{x'_i} = \{(m,\lfloor r \rfloor) \mid a{:}(m,r) \in M', r < \max, \mathit{frct}(r) = x'_i\} = \{(m,\lfloor r \rfloor) \mid a{:}(m,r-d) \in M, r-d < \max, \mathit{frct}(r-d) = x_i\} = A^{x_i}$.
- $A_\infty = \{(m,\max+1) \mid a{:}(m,r) \in M',\ r > \max\} = \{(m,\max+1) \mid a{:}(m,r) \in M,\ r-d > \max\} \cup \{(m,\max+1) \mid a{:}(m,r) \in M,\ r = \max\} = A_\infty + A_0^{=}$.

(5) Suppose that $R_M \overset{\Delta}{\twoheadrightarrow} R'$. As $\overset{\Delta}{\twoheadrightarrow}$ is the reflexive and transitive closure of $\overset{\delta}{\twoheadrightarrow}$, we have that $R_M = R_0 \overset{\delta}{\twoheadrightarrow} R_1 \overset{\delta}{\twoheadrightarrow} \dots \overset{\delta}{\twoheadrightarrow} R' = R_k$. We can prove by an inductive reasoning that for each $i \in n^+$, there exist M_i, d_i such that $R_i = R_{M_i}$ and $M_{i-1} \xrightarrow{d_i} M_i$, by applying the previous claim to each R_i. Therefore, we have that $M = M_0 \xrightarrow{d_1} M_1 \xrightarrow{d_2} \dots \xrightarrow{d_k} M_k$ and $R' = R_k = R_{M_k}$. Therefore, if we take $d = \sum_{i \in k^+} d_i$, then we have that $M \xrightarrow{d} M_k$ and $R' = R_{M_k}$

As $\twoheadrightarrow = \overset{\Delta}{\twoheadrightarrow} \cup \bigcup_{t \in T} \overset{t}{\twoheadrightarrow}$, Prop. 5 easily follows from the previous lemma.

Proposition 3 *Given $p \in P$ we can compute a set of regions $\mathcal{R}_p$ such that there is a reachable marking that marks p iff $\uparrow\mathcal{R}_p$ can be reached.*

Proof. Let $R_0^r = \{(p,r)\} * \emptyset$ for each $r \in \max^*_\infty$, $R_\infty = \emptyset * \{(p,\max+1)\}$ and $R^r = \emptyset * \{(p,r)\} * \emptyset$ for $r \in (\max-1)^*$ Let us see that $\mathcal{R}_p = \{R_0^r \mid r \in \max^*_\infty\} \cup \{R^r \mid r \in (\max-1)^*\} \cup \{R_\infty\}$ satisfies the thesis. First, let us assume that $M_0 \rightarrow^* M$ with $a:(m,r) \in M$ with $p \in m$. By Prop. 5 we have $R_{M_0} \twoheadrightarrow^* R_M = A_0 * A_1 * ... * A_n * A_\infty$. Let us distinguish cases for $r \in \mathbb{R}_{\geq 0}$. If $r \in \max^*$ then by Def. 17, $(m,r) \in A_0$ and $R_0^r \sqsubseteq R_M$. If $r > \max$ also by Def. 17 we have $(m,\max+1) \in A_\infty$, so that $R_\infty \sqsubseteq R_M$. Finally, if $r \leq \max$ and $r \notin \mathbb{N}$, we have $(m,\lfloor r \rfloor) \in A_i$ for some $i \in n^+$, so that $R^{\lfloor r \rfloor} \sqsubseteq R_M$. In any case, $R_M \in \uparrow\mathcal{R}_p$.

Conversely, let us assume that $R_{M_0} \twoheadrightarrow^* R$ with $R \in \uparrow\mathcal{R}_p$. By Prop. 5 there is M reachable such that $R = R_M$. Since $R_M \in \uparrow\mathcal{R}_p$ there is $R' \in \mathcal{R}_p$ such that $R' \sqsubseteq R$. Analogously as in the converse implication, and using again Def. 17, we distinguish cases over R' obtaining in any case that $a:(m,r) \in M$ for some m with $p \in m$, and we conclude.

C2. *ν-lsPN* are Well Structured Transition systems

Proposition$\sqsubseteq$ *is a decidable wpo.*

Proof. In the first place, it is trivially decidable. To prove that it is a wpo, let us remark that a region $R = A_0 * A_1 * ... * A_n * A_\infty$ can be seen as an element of $X = X^{\oplus}_{max^*} \times (X^{\oplus}_{(max-1)^*})^{\circledast} \times X^{\oplus}_{\{max+1\}}$, where for every $I \subseteq (max+1)^*$, $X_I = P^{\oplus} \times I$. Indeed, $A_0 \in X^{\oplus}_{max^*}$, $A_\infty \in X^{\oplus}_{\{max+1\}}$ and $u = A_1 * ... * A_n$ can be seen as a word over $X^{\oplus}_{(max-1)^*}$. Therefore, $\sqsubseteq$ is just the standard order in X, as defined in the preliminaries. Then, $\sqsubseteq$ is a wpo because it is built from wpos (finite sets with equality[8]) using operators that preserve well-orders (multisets, words and the product).

Lemma 5. *If $R_1 \overset{\delta}{\twoheadrightarrow} R_2$ and $R_1 \sqsubseteq R_1'$ then there is R_2' such that $R_1' \sqsubseteq R_2'$ and $R_1' \overset{\Delta}{\twoheadrightarrow} R_2'$.*

Proof. Let $R_1 = A_0 * A_1 * ... * A_n * A_\infty$ and $R_1' = B_0 * u_0 * B_1 * ... * B_n * u_n * B_\infty$ with $A_i \leq^{\oplus} B_i$. First we assume that $A_0 \neq \emptyset$, so that $B_0 \neq \emptyset$. By Def. 18 we have $R_2 = \emptyset * A_0^< * A_1 * ... * A_n * (A_\infty + A_0^=)$ and since $B_0 \neq \emptyset$ we also have $R_1' \overset{\delta}{\twoheadrightarrow} R_2' = \emptyset * B_0^< * u_0 * B_1 * ... * B_n * u_n * (B_\infty + B_0^=)$. Since $A_0 \leq B_0$ we also have $A_0^< \leq B_0^<$, $A_0^= \leq B_0^=$ and thus $(A_\infty + A_0^=) \leq (B_\infty + B_0^=)$. Then $R_1' \sqsubseteq R_2'$.

Now let us assume that $A_0 = \emptyset$, so that $R_2 = A_n^{+1} * A_1 * ... * A_{n-1} * A_\infty$. We also assume that $B_0 \neq \emptyset$ (the other case is only slightly more simple). If $u_n = C_1 * ... * C_k$ then $R_1' \overset{\delta}{\twoheadrightarrow}{}^* R_2' = B_n^{+1} * (C_1^{+1})^< * ... * (C_k^{+1})^< * B_0^< * u_0 * B_1 * u_1 * ... * u_{n-1} * (B_\infty + B_0^= + (C_1^{+1})^= + ... + (C_k^{+1})^=)$ in $2k+2$ steps, and clearly $R_2 \sqsubseteq R_2'$.

Lemma 6. *If $R_1 \overset{\Delta}{\twoheadrightarrow} R_2$ and $R_1 \sqsubseteq R_1'$ then there is R_2' such that $R_1' \sqsubseteq R_2'$ and $R_1' \overset{\Delta}{\twoheadrightarrow} R_2'$.*

Proof. Since $\overset{\Delta}{\twoheadrightarrow}$ is the reflexive and transitive closure of $\overset{\delta}{\twoheadrightarrow}$, it follows from the previous lemma.

Lemma 7. *Let R and R' be two regions such that $R \sqsubseteq R'$. If $A_0 * A_1 * ... * A_n * A_\infty$ is an $\emptyset$-expansion of R, then there is an $\emptyset$-expansion of R' of the form $B_0 * u_0 * B_1 * u_1 * ... * u_{n-1} * B_n * u_n * B_\infty$ such that for all $i \in n_\infty^*$:*

- *$A_i \leq^{\oplus} B_i$,*
- *$(\emptyset, r) \in A_i$ iff $(\emptyset, r) \in B_i$.*

Proof. Indeed, the $\emptyset$-expansion of R' in the lemma can be obtained by adding to R' the same empty instances added to R in order to obtain $A_0 * A_1 * ... * A_n * A_\infty$.

Lemma 8. *If $R_1 \overset{t}{\twoheadrightarrow} R_1'$ and $R_1 \sqsubseteq R_2$ then there is R_2' such that $R_2 \sqsubseteq R_2'$ and $R_2 \overset{t}{\twoheadrightarrow} R_2'$.*

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[8] Multiset containment is the multiset order induced by the equality.

Proof. Assume $R_1 \overset{t}{\twoheadrightarrow} R'_1$ as in Def. 20. By the previous lemma, there is an $\emptyset$-expansion of R_2 of the form $A_0^2 * u_0 * A_1^2 * u_1 * ... * u_{n-1} * A_n^2 * u_n * A_\infty^2$ such that $A_i \leq^\oplus A_i^2$ and $(\emptyset, r) \in A_i$ iff $(\emptyset, r) \in A_i^2$ for all i. Let us see that we can fire t in R_2 as for R_1, also taking $k_0, ..., k_\infty$, ordering $Var(t)$ as $\{x_{ij} \mid i \in n_\infty^*,\ j \in k_i^+\}$ and taking $\overline{A}_i^2 = \{(m_{ij}^2, r_{ij}) \mid j \in k_k^+\} \subseteq A_i^2$ so that $m_{ij} \subseteq m_{ij}^2$.

- If $x_{ij} \in \Upsilon$ then $m_{ij} = \emptyset$, so (previous lemma) $m_{ij}^2 = \emptyset$.
- $F_t(x_{ij}) \subseteq m_{ij} \subseteq m_{ij}^2$.
- r_{ij} are as before, so they satisfy the requirements in the third item.

Then, we can take:

- $B_i^2 = A_i^2 - \overline{A}_i^2$, that satisfies $B_i \leq^\oplus B_i^2$,
- ${m'_{ij}}^2 = (m_{ij}^2 - F_t(x_{ij})) + H_t(x_{ij})$, satisfying $m'_{ij} \subseteq {m'_{ij}}^2$,
- We take r'_{ij} and τ as for R_1.

Then we can define $D_k^2 = \{({m'_{ij}}^2, r'_{ij}) \mid \tau(i,j) = k)\}$, $C_k^2 = B_k^2 + D_k^2$ and $C_{ab}^2 = \{({m'_{ij}}^2, r'_{ij}) \mid \tau(i,j) = (a,b)\}$. Notice that $C_k \leq^\oplus C_k^2$ and $C_{ab} \leq^\oplus C_{ab}^2$. Then $C_0 * C_{01} * ... * C_{0l} * C_1 * ...... * C_{nl} * C_\infty$ is smaller with respect to $\sqsubseteq$ than $C_0^2 * u_0 * C_{01}^2 * ... * C_{0l}^2 * C_1^2 * u_1 * ...... * C_{nl}^2 * C_\infty^2$.[9] Then, the same relation holds between their $\emptyset$-contractions, and taking R'_2 the $\emptyset$-contraction of the latter, the thesis follows.

Notice that, in the previous proof, we could have chosen other ways to fire t in order to reach a region greater than R'_1 from R_2, by choosing other ways in which to insert the "extra" instances in $u_0, ..., u_n$. For instance, we could have taken R'_2 as the $\emptyset$-contraction of $C_0 * u_0 * ... * u_n * C_{01} * ... * C_{0l} * C_1 * ...... * C_{nl} * C_\infty$, in which they are all at the beginning.

Proposition 6 *$\twoheadrightarrow$ is compatible with $\sqsubseteq$.*

Proof. It follows as a corollary of Lemma 6 and Lemma 8.

Now, we prove that the effective *Pre*-basis property holds for .

For that purpose, we need to compute $\min(\uparrow Pre(\uparrow R))$ for any region R. We split *Pre* into $Pre_\Delta(R) = \{R' \mid R' \overset{\Delta}{\twoheadrightarrow} R\}$ and $Pre_t(R) = \{R' \mid R' \overset{t}{\twoheadrightarrow} R\}$, and we we define $\overline{Pre}_\Delta$ and $\overline{Pre}_t$ for each $t \in T$, so that $Pre_\Delta(\uparrow R) = \uparrow \overline{Pre}_\Delta(R)$ and $Pre_t(\uparrow R) = \uparrow \overline{Pre}_t(R)$. First, we define $\overline{Pre}_\Delta$, the function that computes the predecessors corresponding to time delays, using in turn $\overline{Pre}_\delta$ as an auxiliary function, which corresponds to small time delays. Then $\overline{Pre}_\Delta$ is the reflexive and transitive closure of $\overline{Pre}_\delta$.

[9] Notice that, even if strictly speaking they are not regions because they may contain empty instances, they still can be compared with $\sqsubseteq$.

Definition 21 ($\overline{Pre}_\delta$)**.** *Let* $R = A_0 * A_1 * \ldots * A_n * A_\infty$*. We define* $\overline{Pre}_\delta(R)$ *(and extend it pointwise) as*

$$\begin{cases} \{(A_1 + B_0^{-1}) * A_2 * ... * A_n * B_\infty, \\ \quad B_0^{-1} * A_1 * ... * A_n * B_\infty \mid A_\infty = B_0 + B_\infty\} \text{ if } A_0 = \emptyset \\ \\ \{A_1 * A_2 * ... * A_n * A_0^{-1} * A_\infty \mid A_0^{-1} \text{ defined}\}, \text{ otherwise} \end{cases}$$

Let $\overline{Pre}_\delta{}^0(\mathcal{R}) = \mathcal{R}$, $\overline{Pre}_\delta{}^{i+1}(\mathcal{R}) = \overline{Pre}_\delta{}^i(\mathcal{R}) \cup \overline{Pre}_\delta(\overline{Pre}_\delta{}^i(\mathcal{R}))$, and $\overline{Pre}_\Delta(R) = \bigcup_{i \geq 0} \overline{Pre}_\delta{}^i(\{R\})$.

Lemma 9. *Given a region* R*,* $\overline{Pre}_\Delta(R)$ *is finite and* $\uparrow\overline{Pre}_\Delta(R) = Pre_\Delta(\uparrow R)$

We split the previous lemma in the following two lemmas, that we prove separately.

Lemma 10. *Given a region* R*,* $\overline{Pre}_\Delta(R)$ *is finite.*

Proof. For any $R = A_0 * A_1 * ... * A_n * A_\infty$ we define $size(R) = (r, i, |A_\infty|) \in n^* \times n^*_\infty \times \mathbb{N}$, where $(r, i) = \min\{(r, i) \mid (m, r) \in A_i,\ i \in n^*\}$, where the pairs (r, i) are ordered lexicographically, and we also compare tuples $size(R)$ lexicographically. If $size(R) > (0, 0, 0)$ one of the following holds:

- $size(R) = (r, 0, s)$: then $\overline{Pre}_\delta(R) = \{R'\}$ with $R' = \emptyset * A_1 * A_2 * ... * A_n * A_0^{-1} * A_\infty$ and $size(R') = (r-1, n, s)$.
- $size(R) = (r, i, s)$ with $0 < i \leq n$: then the ages in $A_0, ..., A_{i-1}$ are at least $r+1$. The case $A_0 \neq \emptyset$ is analogous to the previous one: R' as in the previous case is the only region in $\overline{Pre}_\delta(R)$, but now $size(R') = (r, i-1, s)$. If $A_0 = \emptyset$ then any R' in $\overline{Pre}_\delta(R)$ is either of the form $(A_1 + B_0^{-1}) * A_2 * ... * A_n * B_\infty$ or $B_0^{-1} * A_1 * ... * A_n * B_\infty$, with $A_\infty = B_0 + B_\infty$, so that $size(R')$ is either $(r, i-1, s')$ in the first case, or (r, i, s') in the second case. Notice also that in the second case, if $R \neq R'$ then $s' < s$.
- If $size(R) = (\max +1, \infty, s)$ then $R = \emptyset * A_\infty$ and every R' in $\overline{Pre}_\delta(R)$ is of the form $R' = B_0^{-1} * B_\infty$ with $A_\infty = B_0 + B_\infty$. Notice that if $B_0 = \emptyset$ then $R = R'$. Otherwise, $size(R') = (\max, 0, s')$.
- If $size(R) = (0, 0, s)$ then A_0^{-1} is undefined, and $\overline{Pre}_\delta(R) = \emptyset$.

Assume by contradiction that $\overline{Pre}_\Delta(R)$ is infinite. Then there is a sequence $(R_i)_{i \geq 0}$ of pairwise different regions such that $R_{i+1} \in \overline{Pre}_\delta(R_i)$. By the previous items notice that $size(R_{i+1}) < size(R_i)$, which is a contradiction because the lexicographic order is well-founded in $n^* \times n^*_\infty \times \mathbb{N}$.

Lemma 11. *Given a region* R*,* $\uparrow\overline{Pre}_\Delta(R) = Pre_\Delta(\uparrow R)$

Proof. Let us first see that $Pre_\Delta(\uparrow R) \subseteq \uparrow\overline{Pre}_\Delta(R)$, for which it is enough to see that $Pre_\delta(\uparrow R) \subseteq \uparrow\overline{Pre}_\Delta(R)$. Let $R = A_0 * A_1 * ... * A_n * A_\infty$, R' and R'' such that $R'' \xrightarrow{\delta} R'$ with $R \sqsubseteq R'$. Since $R \sqsubseteq R'$ we can write $R' = B_0 * u_0 * B_1 * ... * B_n * u_n * B_\infty$ with $A_i \leq^\oplus B_i$. We distinguish three cases: (i) If $A_0 \neq \emptyset$ then $B_0 \neq \emptyset$, in which case $R'' = \emptyset * u_0 * B_1 * ... * B_n * u_n * B_0^{-1} * B_\infty$, which is greater than $\emptyset * A_1 * ... * A_n * A_0^{-1} * A_\infty \in \overline{Pre}_\delta(R) \subseteq \overline{Pre}_\Delta(R)$. (ii) If $A_0 = \emptyset$ and $B_0 \neq \emptyset$ then $R'' = \emptyset * u_0 * B_1 * ... * B_n * u_n * B_0^{-1} * B_\infty \in \uparrow R \subseteq \uparrow \overline{Pre}_\Delta(R)$. (iii) Finally, if $A_0 = B_0 = \emptyset$ we distinguish two subcases. If $u_0 = \epsilon$ then $R'' = (B_1 + C_1) * u_1 * B_2 * ... * B_n * u_n * C_2$ with $C_1^{+1} + C_2 = B_\infty$, which is greater than $(A_1 + D_1) * A_2 * ... * A_n * D_2 \in \overline{Pre}_\delta(R)$ for some $D_1^{+1} + D_2 = A_\infty$. If $u_0 \neq \epsilon$ then $u_0 = B * u_0'$, in which case $R'' = (B + C_1) * u_0' * B_1 * ... * B_n * u_n * C_2$ with $C_1^{+1} + C_2 = B_\infty$, which is greater than $D_1 * A_1 * ... * A_n * D_2 \in \overline{Pre}_\delta(R)$ for some $D_1^{+1} + D_2 = A_\infty$. For the other containment, it is enough to see that $\uparrow \overline{Pre}_\delta(R) \subseteq Pre_\Delta(\uparrow R)$. Notice that for any $R' \in \overline{Pre}_\delta(R)$ we have $R' \xrightarrow{\Delta} R$. Hence, given $R'' \in \uparrow \overline{Pre}_\delta(R)$ we have $R' \sqsubseteq R''$ for some $R' \in \overline{Pre}_\delta(R)$ such that $R' \xrightarrow{\Delta} R$. Then, by compatibility of $\xrightarrow{\delta}$ (see Lemma 5), $R'' \xrightarrow{\Delta} \uparrow R$, so that $R'' \in Pre_\Delta(\uparrow R)$.

Now, we define $\overline{Pre}_t$ to compute the predecessors corresponding to firings of transitions. We first define for $t \in T$ and each region R a family $\mathcal{F}(t, R)$ of functions which assign to each variable in $Var(t)$ a part of the region taking part of the firing. Then we define $\overline{Pre}_t(R)$ considering every $f \in \mathcal{F}(t, R)$.

Definition 22 **($\mathcal{F}(t, R)$).** *Let $t \in T$ and a region $R = A_0 * A_1 * \ldots * A_n * A_\infty$, with $A_i = \{(m_{i1}, r_{i1}), \ldots, (m_{ik_i}, r_{ik_i})\}$. Suppose that $l = |Var(t)|$ and $q = \max\{k_i \mid i \in n^*_\infty\}$. A function $f : Var(t) \to (n+1)^*_\infty \times (q+1)^* \times (max+1)^* \times (n^* \cup \{\infty\}) \times l^*$ is in $\mathcal{F}(t, R)$ iff for all $f(x) = (b_1, b_2, b_3, b_4, b_5)$:*

- *If $\mathcal{G}_t^2(x) = ro$, $b_1 \in n^+$ and $b_2 \leq k_{b_1}$ then $b_3 = r_{b_1 b_2}$, $b_4 = b_1$ and $b_5 = 0$.*
- *If $b_1 \in n^+$ and $b_2 \leq k_{b_1}$ then $r_{b_1 b_2} + 0.5 \in \mathcal{G}_t^2(x)$.*
- *If $b_1 = 0$ and $b_2 \leq k_0$ then $r_{b_1 b_2} \in \mathcal{G}_t^2(x)$.*
- *If $x \in \Upsilon$ then $m_{b_1 b_2} \leq H_t(x)$.*
- *If $b_1 = \infty$ then $\max + 0.5 \in \mathcal{G}_t^2(x)$.*
- *If $b_4 \in n^+$ then $b_3 + 0.5 \in \mathcal{G}_t^1(x)$.*
- *If $b_4 = 0$ then $b_3 \in \mathcal{G}_t^1(x)$.*
- *If $b_4 = \infty$ iff $b_3 = max + 1$.*
- *If $y \neq x$ and $f(y) = (b_1', b_2', b_3', b_4', b_5')$, then $(b_1, b_2) \neq (b_1', b_2')$.*

Intuitively, the first two numbers that the previous functions assign to a variable x, correspond to the selection of the part of the region we assign to x to remove the effects of H_t. Analogously, the two last components manage the effects of F_t. The third number assigns to each variable the natural number that correspond to the age of the instance in the predecessor.

Clearly, the family $\mathcal{F}(t, R)$ is finite. We define $\overline{Pre}_t(R)$ as the effects of computing the predecessors of R according to all the functions in $\mathcal{F}(t, R)$.

Definition 23 **($\overline{Pre}_t$).** *Let $l = |Var(t)|$. Given $t \in T$, $f \in \mathcal{F}(t, R)$ and $R = A_0 * A_1 * \ldots * A_n * A_\infty$, with $A_i = \{(m_{i1}, r_{i1}), ..., (m_{ik}, r_{ik})\}$, we define $\overline{Pre}_{ft}(R)$ as follows:*

- *First, we define $R'' = A_{00}' * A_{01}' * \ldots * A_{0l}' * A_{10}' * A_{11}' * ... * A_{nl}' * A_{\infty 0}'$, where:*
  - *$A_{j0}' = A_j - \{(m_{jk}, r_{jk}) \mid \exists x$ with $f(x) = (j, k, b_3, b_4, b_5)$ for some $b_3, b_4, b_5\}$*

- $A'_{\infty 0} = A_\infty - \{(m_{\infty k}, max{+}1) \mid \exists x$ *with* $f(x) = (\infty, k, b_3, b_4, b_5)$ *for some* $b_3, b_4, b_5\}$
- $A'_{ij} = \emptyset$ *elsewhere.*

– *For each* $x \in Var(t)$, *if* $f(x) = (b_1, b_2, b_3, b_4, b_5)$, *then we define* $m'_x = (m_{b_1 b_2} \ominus H_t(x)) + F_t(x)$ *and* $r'_x = b_3$, *where* $(m_1 \ominus m_2)(x) = \max(0, m_1(x) - m_2(x))$.

– *Finally,* $\overline{Pre}_{ft}(R)$ *is the* $\emptyset$*-contraction of* $B_{00} * B_{01} * \ldots * B_{0l} * B_{10} * B_{11} * \ldots * B_{nl} * B_{\infty 0}$, *where for each* $i \in n^*_\infty$ *and* $j \in l^*$, $B_{ij} = A'_{ij} + \{(m'_x, r'_x) \mid f(x) = (b_1, b_2, b_3, i, j)$ *for some* $b_1, b_2, b_3\}$.

Then, we define $\overline{Pre}_t(R) = \{\overline{Pre}_{ft}(R) \mid f \in \mathcal{F}(t, R)\}$.

Intuitively, in R'' we have removed the instances corresponding to the effects of H_t, and added l empty multisets of instances between each A_i and A_{i+1} in order to be able to add tokens with new fractional parts as predecessors.

Lemma 2 *Given a region* R, *we can compute a finite set* $\overline{Pre}_t(R)$ *such that* $Pre_t(\uparrow R) = \uparrow\overline{Pre}_t(R)$

Proof. Clearly, $\overline{Pre}_t(R)$ as defined above is computable. First, we prove $\uparrow\overline{Pre}_t(R) \subseteq Pre_t(\uparrow R)$. Let $R' \in \uparrow\overline{Pre}_t(R)$, we are going to prove that $R' \in Pre_t(\uparrow R)$. Therefore, we need to prove that there is R''' with $R \sqsubseteq R'''$ such that $R' \xrightarrow{t} R'''$. Let us call $R = A_0 * A_1 * ... * A_{n_A} * A_\infty$ with $A_i = \{(m^A_{ij}, r^A_{ij}) \mid j \in |A_i|^+\}$ and $R' = E_0 * E_1 * ... * E_{n_E} * E_\infty$ with $E_i = \{(m^E_{ij}, r^E_{ij}) \mid j \in |E_i|^+\}$

First, let us prove that t is enabled at R'. As $R' \in \uparrow\overline{Pre}_t(R)$, there is $R'' \in \overline{Pre}_t(R)$ such that $R'' \sqsubseteq R'$. Then, there is $f \in \mathcal{F}(t, R)$ such that $R'' = \overline{Pre}_{ft}(R)$. Suppose that $B_{00} * B_{01} * \ldots * B_{0l} * B_{10} * B_{11} * \ldots * B_{nl} * B_{\infty 0}$, is the $\emptyset$-expansion of R'' obtained in the definition of $\overline{Pre}_{ft}(R)$. Then, for each $x \in Var(t)$, if $f(x) = (b_1, b_2, b_3, b_4, b_5)$, there is a marking $(m^x, r^x) \in B_{b_4 b_5}$ (corresponding to (m'_x, r'_x) in the definition of $\overline{Pre}_{ft}(R)$) such that $F_t(x) \subseteq m^x$ and $r^x = b_3$. As $R'' \sqsubseteq R'$, for each x, there is (m^x_1, r^x) in some E_i of R' such that $F_t(x) \subseteq m^x \subseteq m^x_1$. Moreover, by the properties required for b_3 in the definition of $\mathcal{F}(t, R)$, $r^x = b_3 \in \mathcal{G}^1_t(x)$ if $i \in \{0, \infty\}$, and $r^x + 0.5 = b_3 + 0.5 \in \mathcal{G}^1_t(x)$, otherwise. Therefore, t is enabled at R'.

Now we define the proper τ and a renaming of $Var(t)$, in order to fire t from R'' in a way such that we obtain a region R''' such that $R \sqsubseteq R'''$. Let us suppose that $R'' = G_0 * G_1 * ... * G_{n_G} * G_\infty$, with $G_i = \{(m^G_{ij}, r^G_{ij}) \mid j \in |G_i|^+\}$, such that there is a strictly increasing $c : (n_G)^*_\infty \to (n_E)^*_\infty$ such that for each $i \in (n_G)^*_\infty$, $G_i \leq^\oplus E_{c(i)}$ and in particular, for each $j \in |G_i|^+$, $(m^G_{ij}, r^G_{ij}) \leq (m^E_{ij}, r^E_{ij})$.

Then, if $f(x) = (b_1, b_2, b_3, b_4, b_5)$ we define a renaming $x = x_{c(i)j}$ such that $B_{b_4 b_5} = G_i$ and $(m^G_{ij}, r^G_{ij}) = ((m^A_{b_1 b_2} \ominus H_t(x) + F_t(x)), b_3)$. Moreover, we define $r'_{c(i)j} = r^A_{b_1 b_2}$. We define τ of the following form:

– If $B_{b_1 0} \neq \emptyset$ and $B_{b_1 0} = G(i)$ then $\tau(x) = c(i)$.

– If $B_{b_1 0} = \emptyset$, suppose that $B_{i_1 i_2}$ is the greater set lower than $B_{b_1 0}$ if we consider the lexicographic order, such that $B_{i_1 i_2} \neq \emptyset$. Moreover, suppose that between $B_{i_1 i_2}$ and $B_{b_1 b_2}$ there are k sets and $B_{i_1 i_2} = G_i$. Then, we define $\tau(x) = (c(i), k)$

Let us see that if $R' \stackrel{t}{\twoheadrightarrow} R'''$ with the previous renaming and τ, given A_i of R, there is D of R''' with $A_i \leq^{\oplus} D$.

Suppose $(m^A_{ij}, r^A_{ij}) \in A_i$. We distiguish two cases, depending on whether $B_{i0} = \emptyset$ or not. First, we consider $B_{i0} \neq \emptyset$. There are several possibilities:

- If there is no $x \in Var(t)$ with $f(x) = (i, j, b_3, b_4, b_5)$ then $(m^A_{ij}, r^A_{ij}) \in B_{i0} = G_k \leq^{\oplus} E_{c(k)}$ for some k. Then, because of how we have defined the renaming and τ, we have that there is (m', r^A_{ij}) with $(m^A_{ij}, r^A_{ij}) \leq (m', r^A_{ij}) \in C_{c(k)}$, where $C_{c(k)}$ is in the $\emptyset$-expansion of R''' as in the definition of firing for $R' \stackrel{t}{\twoheadrightarrow} R'''$.
- Now, suppose there is $x \in Var(t)$ with $f(x) = (i, j, b_3, b_4, b_5)$. Then, $(m', b_3) \in B_{ij} = G_{h_1}$ for some h_1, where $m' = (m \ominus H_t(x)) + F_t(x)$. Suppose that $(m', b_3) = (m^G_{h_1h_2}, b_3)$. Then, we know that $(m', b_3) = (m^G_{h_1h_2}, b_3) \leq (m^E_{c(h_1)h_2}, b_3) \in E_{c(h_1)}$. As we suppose $B_{b_10} \neq \emptyset$, we can consider that $B_{i0} = G_k \leq^{\oplus} E_{c(k)}$, for the same k as in the previous case. Then, because of how we have defined the renaming and τ, we know that $x = x_{kh_2}$, $r'_{kh_2} = r^A_{ij}$ and $\tau(x) = c(k)$. Therefore, if we call $m' = (m^G_{h_1h_2} - F_t(x)) + H_t(x)$, then we know that $(m^A_{ij}, r^A_{ij}) \leq (m', r^A_{ij}) \in C_{c(k)}$, where $C_{c(k)}$ is in the $\emptyset$-expansion of R''' as in Def. 20.

As the renaming and τ we have defined are injective, we have that for each $(m^A_{ij}, r^A_{ij}) \in A_i$ in this case, we have a different $(m, r) \in C_{c(k)}$ such that $(m^A_{ij}, r^A_{ij}) \leq (m, r)$.

Now, we suppose $B_{i0} = \emptyset$. Then we know that there is $x \in Var(t)$ with $f(x) = (i, j, b_3, b_4, b_5)$. Therefore, is $B_{i_1i_2}$ is the greater set lower than B_{i0} if we consider the lexicographic order, such that $B_{i_1i_2} \neq \emptyset$; between $B_{i_1i_2}$ and B_{i0} there are k sets and $B_{i_1i_2} = G_i$, then $\tau(x) = (c(i), k)$, if in the renaming $x = x_{h_1h_2}$ then $G_{h_1h_2} = \emptyset$ and $r'(x) = r^A_{ij}$. Therefore, is $m' = H_t(x)$ then $(m^A_{ij}, r^A_{ij}) \leq (m', r^A_{ij}) \in C_{(c(i),k)}$, where $C_{(c(i),k)}$ is in the $\emptyset$-expansion of R''' as in the definition of firing for $R' \stackrel{t}{\twoheadrightarrow} R'''$. Again, as the renaming and τ we have defined are injective, we have that for each $(m^A_{ij}, r^A_{ij}) \in A_i$ in this case, we have a different $(m, r) \in C_{(c(i),k)}$ such that $(m^A_{ij}, r^A_{ij}) \leq (m, r)$, and we are done.

Now, we prove that $Pre_t(\uparrow R) \subseteq \uparrow \overline{Pre}_t(R)$. Let $R = A_0 * A_1 * ... * A_n * A_\infty$, R' and R'' such that $R'' \stackrel{t}{\twoheadrightarrow} R'$ with $R \sqsubseteq R'$. Since $R \sqsubseteq R'$ we can write $R' = E_0 * u_0 * E_1 * ... * E_n * u_n * E_\infty$, with $A_i \leq^{\oplus} E_i$. Let us call $A_i = \{(m^A_{ij}, r^A_{ij}) \mid j \in |A_i|^+\}$ and $E_i = \{(m^E_{ij}, r^E_{ij}) \mid j \in |E_i|^+\}$. Then, we suppose that for each $i \in n^*_\infty, j \in |A_i|$, $(m^A_{ij}, r^A_{ij}) \leq (m^E_{ij}, r^E_{ij})$.

It is enough to see that there exist $f \in \mathcal{F}(t, R')$ such that $R'' \in \Uparrow(\overline{Pre}_{ft}(R))$. Let us define such an f. For that purpose, we need to define several auxiliary functions.

Suppose that τ is the function defined in the firing $R'' \stackrel{t}{\twoheadrightarrow} R'$ and (m'_{ij}, r'_{ij}) are the pairs built like in the definition of firings of transitions for regions for $R'' \stackrel{t}{\twoheadrightarrow} R'$. Let $x_{ij} \in Var(t)$ and $\tau(x_{ij}) = e$. Consider C_e as defined in the Def. 20

for the firing $R'' \xrightarrow{t} R'$. We define a function $\bar{\tau}$, which we will use to define the two first components of f, in the following way:

- If $C_e = E_{e'}$ (which corresponds to $A_{e'}$) and $(m'_{ij}, r'_{ij}) = (m^E_{e'g}, r^E_{e'g})$, $g \leq |A_{e'}|$ then $\bar{\tau}(x_{ij}) = (e', g)$.
- If $C_e = E_{e'}$ (which corresponds to $A_{e'}$) and $g > |A_{e'}|$ then $\bar{\tau}(x_{ij}) = (e', |A_{e'}| + 1)$.
- $\bar{\tau}(x_{ij}) = (n+1, 1)$ otherwise.

Now we define a function τ' in order to define the two last components of f, the following way: If $C_i = E_{i'}$ and $j \leq |A_{i'}|$ then $\tau'(x_{ij}) = (i', 0)$. Otherwise, we define $\tau'(x_{ij})$ in any way such that if $\tau'(x_{i_1 j_1}) = (a_1, b_1)$, $\tau'(x_{i_2 j_2}) = (a_2, b_2)$ and $i_1 > i_2$ then $(a_1 > a_2) \vee (a_1 = a_2 \wedge b_1 > b_2)$ and if $i_1 = i_2$ then $(a_1, b_1) = (a_2, b_2)$.

Then, we define $f(x_{ij}) = (b_1, b_2, r_{ij}, b_4, b_5)$ such that $\bar{\tau}(b_1, b_2) = (i, j)$ and $\tau'(x_{ij}) = (b_4, b_5)$. From the definition of firing of transitions for regions, it can be seen that $f \in \mathcal{F}(t, R')$.

Now, we have to see that $R'' \in \Uparrow(\overline{Pre}_{ft}(R))$, or equivalently, that $\overline{Pre}_{ft}(R) \sqsubseteq R''$. We show that given a word D'_i of $\overline{Pre}_{ft}(R)$ which corresponds to a $B_{i_1 0}$ of the $\emptyset$-expansion of $\overline{Pre}_{ft}(R)$ obtained in the definition of $\overline{Pre}_{ft}$, there is a word D_j of R'' such that $D'_i \leq^{\oplus} D_j$ (the rest of the cases are easier). $D'_i = B_{i_1 0} = A'_{i_1} + \{(m'_x, b_3) \mid f(x) = (b_1, b_2, b_3, i_1, 0)$ for some $b_1, b_2, b_3\}$ where $A'_{i_1} = A_{i_1} - \{(m_{i_1 i_2}, r_{i_1 i_2}) \in A_{i_1} \mid \exists x$ with $f(x) = (i_1, i_2, b_3, b_4, b_5)$ for some $b_3, b_4, b_5\}$ and $m'_x = m^A_{b_1 b_2} \ominus H_t(x)) + F_t(x)$.

Because of how we have defined f, $A'_{i_1} = A_{i_1} - \{(m_{i_1 i_2}, r_{i_1 i_2}) \in B_{i_1} \mid \exists x$ with $f(x) = (i_1, i_2, b_3, b_4, b_5)$ for some $b_3, b_4, b_5\} \leq^{\oplus} E_{i_1} - \{(m_{i_1 i_2}, r_{i_1 i_2}) \in E_{i_1} \mid \exists x$ with $f(x) = (i_1, i_2, b_3, b_4, b_5)$ for some $b_3, b_4, b_5\} = E_{i_1} - \{(m_{i_1 i_2}, r_{i_1 i_2}) \in E_{i_1} \mid \exists x$ with $\tau(x) = i_1, (m_{i_1 i_2}, r_{i_1 i_2}) = (m'_{ij}, r'_{ij})\}$, where (m'_{ij}, r'_{ij}) is like in Def. 20. Intuitively, this means that B'_{i_1} is contained in the set obtained from E_{i_1} by removing the set of pairs affected by the effects of H_t.

Moreover, if $x \in Var(t)$ and $f(x) = (b_1, b_2, b_3, i_1, 0)$ then $m'_x = (m^A_{b_1 b_2} \ominus H_t(x)) + F_t(x) \leq m^E_{b_1 b_2} \ominus H_t(x)) + F_t(x)$. Therefore, $\{(m'_x, b_3) \mid f(x) = (b_1, b_2, b_3, i_1, 0)$ for some $b_1, b_2, b_3\} \leq \{(m', b_3) \mid f(x) = (b_1, b_2, b_3, i_1, 0)$ for some $b_1, b_2, b_3, m' = m^E_{b_1 b_2} \ominus H_t(x)) + F_t(x)\}$.

Therefore, we have $D'_i = B_{i_1 0} = A'_{i_1} + \{(m'_x, b_3) \mid f(x) = (b_1, b_2, b_3, i_1, 0)$ for some $b_1, b_2, b_3\} \leq^{\oplus} E_{i_1} - \{(m_{i_1 i_2}, r_{i_1 i_2}) \in E_{i_1} \mid \exists x$ with $\tau(x) = i_1, (m_{i_1 i_2}, r_{i_1 i_2}) = (m'_{ij}, r'_{ij})$ like in def. of firing$)\} + \{(m', b_3) \mid f(x) = (b_1, b_2, b_3, i_1, 0)$ for some $b_1, b_2, b_3, m' = m^E_{b_1 b_2} \ominus H_t(x)) + F_t(x)\} = D$. Moreover, from the definition of firings of transitions, it can be seen that $D \in R''$. It is easy to see that the D_i are correctly ordered.

Proposition 6. $\min(\uparrow Pre(\uparrow R))$ *is computable for any R.*

Proof. Indeed, by the two previous lemmas, we can compute it as $\min(\overline{Pre}_{\Delta}(R) \cup \bigcup_{t \in T} \overline{Pre}_t(R))$.

Then, we obtain the following result:

Corollary 2. *Control-state reachability is decidable for ν-lsPN.*

Proof. We have proved that the transition system over regions is a WSTS, so coverability is decidable. Since the control-state reachability problem can be reduced to coverability (Prop. 3) the thesis follows.